CALCULUS MADE EASY

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CALCULUS MADE EASY:

BEING A VERY-SIMPLEST INTRODUCTION TO THOSE BEAUTIFUL METHODS OF RECKONING WHICH ARE GENERALLY CALLED BY THE TERRIFYING NAMES OF THE

DIFFERENTIAL CALCULUS

AND THE

INTEGRAL CALCULUS.

BY

SILVANUS P. THOMPSON, F.R.S.

SECOND EDITION, ENLARGED

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What one fool can do, another can.

(Ancient Simian Proverb)

PREFACE TO THE SECOND EDITION.

THE surprising success of this work has led the author to add a considerable number of worked examples and exercises. Advantage has also been taken to enlarge certain parts where experience showed that further explanations would be useful.

The author acknowledges with gratitude many valuable suggestions and letters received from teachers, students, and—critics.

October, 1914.

CONTENTS.

PROLOGUE.

CONSIDERING how many fools can calculate, it is surprising that it should be thought either a difficult or a tedious task for any other fool to learn how to master the same tricks.

Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the text-books of advanced mathematics—and they are mostly clever fools—seldom take the trouble to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way.

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can.

CHAPTER I.

TO DELIVER YOU FROM THE PRELIMINARY TERRORS.

THE preliminary terror, which chokes off most fifth-form boys from even attempting to learn how to calculate, can be abolished once for all by simply stating what is the meaning—in common-sense terms—of the two principal symbols that are used in calculating.

These dreadful symbols are:

(1) d which merely means "a little bit of."

Thus dx means a little bit of x; or du means a little bit of u. Ordinary mathematicians think it more polite to say "an element of," instead of "a little bit of." Just as you please. But you will find that these little bits (or elements) may be considered to be indefinitely small.

(2) $\int$ which is merely a long S, and may be called (if you like) "the sum of."

Thus $\int dx$ means the sum of all the little bits of x; or $\int dt$ means the sum of all the little bits of t. Ordinary mathematicians call this symbol "the integral of." Now any fool can see that if x is considered as made up of a lot of little bits, each of which is called dx, if you add them all up together you get the sum of all the dx's, (which is the same thing as the whole of x). The word "integral" simply means "the whole." If you think of the duration of time for one hour, you may (if you like) think of it as cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

When you see an expression that begins with this terrifying symbol, you will henceforth know that it is put there merely to give you instructions that you are now to perform the operation (if you can) of totalling up all the little bits that are indicated by the symbols that follow.

That's all.

CHAPTER II.

ON DIFFERENT DEGREES OF SMALLNESS.

WE shall find that in our processes of calculation we have to deal with small quantities of various degrees of smallness.

We shall have also to learn under what circumstances we may consider small quantities to be so minute that we may omit them from consideration. Everything depends upon relative minuteness.

Before we fix any rules let us think of some familiar cases. There are 60 minutes in the hour, 24 hours in the day, 7 days in the week. There are therefore 1440 minutes in the day and 10080 minutes in the week.

Obviously 1 minute is a very small quantity of time compared with a whole week. Indeed, our forefathers considered it small as compared with an hour, and called it "one minùte," meaning a minute fraction—namely one sixtieth—of an hour. When they came to require still smaller subdivisions of time, they divided each minute into 60 still smaller parts, which, in Queen Elizabeth's days, they called "second minùtes" (*i.e.*, small quantities of the second order of minuteness). Nowadays we call these small quantities of the second order of smallness "seconds." But few people know *why* they are so called.

Now if one minute is so small as compared with a whole day, how much smaller by comparison is one second!

Again, think of a farthing as compared with a sovereign: it is worth only a little more than $\frac{1}{1000}$ part. A farthing more or less is of precious little importance compared with a sovereign: it may certainly be regarded as a *small* quantity. But compare a farthing with £1000: relatively to this greater sum, the farthing is of no more importance than $\frac{1}{1000}$ of a farthing would be to a sovereign. Even a golden sovereign is relatively a negligible quantity in the wealth of a millionaire.

Now if we fix upon any numerical fraction as constituting the proportion which for any purpose we call relatively small, we can easily state other fractions of a higher degree of smallness. Thus if, for the purpose of time, $\frac{1}{60}$ be called a *small* fraction, then $\frac{1}{60}$ of $\frac{1}{60}$ (being a *small* fraction of a *small* fraction) may be regarded as a *small quantity of the second order* of smallness.*

Or, if for any purpose we were to take 1 per cent. (*i.e.*, $\frac{1}{100}$) as a *small* fraction, then 1 per cent. of 1 per cent. (*i.e.*, $\frac{1}{10,000}$) would be a small fraction of the second order of smallness; and $\frac{1}{1,000,000}$ would

* The mathematicians talk about the second order of "magnitude" (*i.e.* greatness) when they really mean second order of *smallness*. This is very confusing to beginners.

be a small fraction of the third order of smallness, being 1 per cent. of 1 per cent. of 1 per cent.

Lastly, suppose that for some very precise purpose we should regard $\frac{1}{1,000,000}$ as "small." Thus, if a first-rate chronometer is not to lose or gain more than half a minute in a year, it must keep time with an accuracy of 1 part in 1,051,200. Now if, for such a purpose, we regard $\frac{1}{1,000,000}$ (or one millionth) as a small quantity, then $\frac{1}{1,000,000}$ of $\frac{1}{1,000,000}$, that is, $\frac{1}{1,000,000,000,000}$ (or one billionth) will be a small quantity of the second order of smallness, and may be utterly disregarded, by comparison.

Then we see that the smaller a small quantity itself is, the more negligible does the corresponding small quantity of the second order become. Hence we know that *in all cases we are justified in neglecting the small quantities of the second—or third* (or higher)—*orders,* if only we take the small quantity of the first order small enough in itself.

But it must be remembered that small quantities, if they occur in our expressions as factors multiplied by some other factor, may become important if the other factor is itself large. Even a farthing becomes important if only it is multiplied by a few hundred.

Now in the calculus we write dx for a little bit of x. These things such as dx, and du, and dy, are called "differentials," the differential of x, or of u, or of y, as the case may be. [You *read* them as *dee-eks,* or *dee-you,* or *dee-wy.*] If dx be a small bit of x, and relatively small of itself, it does not follow that such quantities as $x \cdot dx$, or $x^2 dx$, or $a^x dx$ are negligible. But $dx \times dx$ would be negligible, being a small quantity of the second order.

A very simple example will serve as illustration.

Let us think of x as a quantity that can grow by a small amount so as to become $x+dx$, where dx is the small increment added by growth. The square of this is $x^2+2x \cdot dx+(dx)^2$. The second term is not negligible because it is a first-order quantity; while the third term is of the second order of smallness, being a bit of a bit of x. Thus if we took dx to mean numerically, say, $\frac{1}{60}$ of x, then the second term would be $\frac{2}{60}$ of x^2, whereas the third term would be $\frac{1}{3600}$ of x^2. This last term is clearly less important than the second. But if we go further and take dx to mean only $\frac{1}{1000}$ of x, then the second term will be $\frac{2}{1000}$ of x^2, while the third term will be only $\frac{1}{1,000,000}$ of x^2.

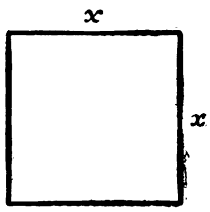

FIG. 1.

Geometrically this may be depicted as follows: Draw a square (Fig. 1) the side of which we will take to represent x. Now suppose the square *to grow* by having a bit dx added to its size each way. The enlarged square is made up of the original square x^2, the two rectangles at the top and on the right, each of which is of area $x \cdot dx$ (or together $2x \cdot dx$), and the little square at the top right-hand corner which is $(dx)^2$. In Fig. 2 we have taken dx as

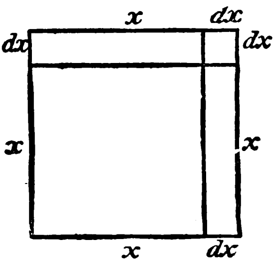

FIG. 2.

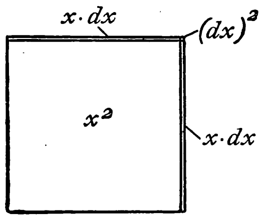

FIG. 3.

quite a big fraction of x—about $\frac{1}{5}$. But suppose we had taken it only $\frac{1}{100}$—about the thickness of an inked line drawn with a fine pen. Then the little corner square will have an area of only $\frac{1}{10,000}$ of x^2, and be practically invisible. Clearly $(dx)^2$ is negligible if only we consider the increment dx to be itself small enough.

Let us consider a simile.

Suppose a millionaire were to say to his secretary: next week I will give you a small fraction of any money that comes in to me. Suppose that the secretary were to say to his boy: I will give you a small fraction of what I get. Suppose the fraction in each case to be $\frac{1}{100}$ part. Now if Mr. Millionaire received during the next week £1000, the secretary would receive £10 and the boy 2 shillings. Ten pounds would be a small quantity compared with £1000; but two shillings is a small small quantity indeed, of a very secondary order. But what would be the disproportion if the fraction, instead of being $\frac{1}{100}$, had been settled at $\frac{1}{1000}$ part? Then, while Mr. Millionaire got his £1000, Mr. Secretary would get only £1, and the boy less than one farthing!

The witty Dean Swift* once wrote:

> "So, Nat'ralists observe, a Flea
> "Hath smaller Fleas that on him prey.
> "And these have smaller Fleas to bite 'em,
> "And so proceed *ad infinitum*."

An ox might worry about a flea of ordinary size—a small creature of the first order of smallness. But he would probably not trouble himself about a flea's flea; being of the second order of smallness, it would be negligible. Even a gross of fleas' fleas would not be of much account to the ox.

* *On Poetry: a Rhapsody* (p. 20), printed 1733—usually misquoted.

CHAPTER III.

ON RELATIVE GROWINGS.

All through the calculus we are dealing with quantities that are growing, and with rates of growth. We classify all quantities into two classes: *constants* and *variables*. Those which we regard as of fixed value, and call *constants*, we generally denote algebraically by letters from the beginning of the alphabet, such as a, b, or c; while those which we consider as capable of growing, or (as mathematicians say) of "varying," we denote by letters from the end of the alphabet, such as x, y, z, u, v, w, or sometimes t.

Moreover, we are usually dealing with more than one variable at once, and thinking of the way in which one variable depends on the other: for instance, we think of the way in which the height reached by a projectile depends on the time of attaining that height. Or, we are asked to consider a rectangle of given area, and to enquire how any increase in the length of it will compel a corresponding decrease in the breadth of it. Or, we think of the way in which any variation in the slope of a ladder will cause the height that it reaches, to vary.

Suppose we have got two such variables that depend one on the other. An alteration in one will bring about an alteration in the other, *because* of this dependence. Let us call one of the variables x, and the other that depends on it y.

Suppose we make x to vary, that is to say, we either alter it or imagine it to be altered, by adding to it a bit which we call dx. We are thus causing x to become $x+dx$. Then, because x has been altered, y will have altered also, and will have become $y+dy$. Here the bit dy may be in some cases positive, in others negative; and it won't (except very rarely) be the same size as dx.

Take two examples.

(1) Let x and y be respectively the base and the height of a right-angled triangle (Fig. 4), of which

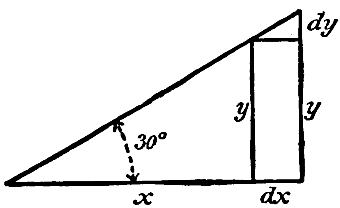

FIG. 4.

the slope of the other side is fixed at 30°. If we suppose this triangle to expand and yet keep its angles the same as at first, then, when the base grows so as to become $x+dx$, the height becomes $y+dy$. Here, increasing x results in an increase of y. The little triangle, the height of which is dy, and the base of which is dx, is similar to the original triangle; and it is obvious that the value of the ratio $\frac{dy}{dx}$ is the same as that of the ratio $\frac{y}{x}$. As the angle is 30° it will be seen that here

$$\frac{dy}{dx}=\frac{1}{1{\cdot}73}.$$

(2) Let x represent, in Fig. 5, the horizontal distance, from a wall, of the bottom end of a ladder,

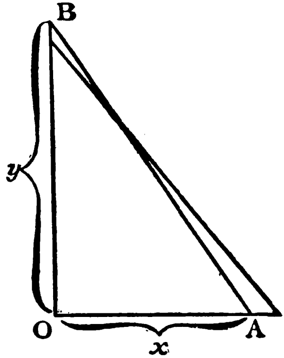

FIG. 5.

AB, of fixed length; and let y be the height it reaches up the wall. Now y clearly depends on x. It is easy to see that, if we pull the bottom end A a bit further from the wall, the top end B will come down a little lower. Let us state this in scientific language. If we increase x to $x+dx$, then y will become $y-dy$; that is, when x receives a positive increment, the increment which results to y is negative.

Yes, but how much? Suppose the ladder was so long that when the bottom end A was 19 inches from the wall the top end B reached just 15 feet from the ground. Now, if you were to pull the bottom end out 1 inch more, how much would the top end come down? Put it all into inches: $x=19$ inches, $y=180$ inches. Now the increment of x which we call dx, is 1 inch: or $x+dx=20$ inches.

How much will y be diminished? The new height will be $y-dy$. If we work out the height by Euclid I. 47, then we shall be able to find how much dy will be. The length of the ladder is

$$\sqrt{(180)^2+(19)^2}=181 \text{ inches.}$$

Clearly then, the new height, which is $y-dy$, will be such that

$$(y-dy)^2=(181)^2-(20)^2=32761-400=32361,$$

$$y-dy=\sqrt{32361}=179{\cdot}89 \text{ inches.}$$

Now y is 180, so that dy is $180-179{\cdot}89=0{\cdot}11$ inch.

So we see that making dx an increase of 1 inch has resulted in making dy a decrease of 0·11 inch.

And the ratio of dy to dx may be stated thus:

$$\frac{dy}{dx}=-\frac{0{\cdot}11}{1}.$$

It is also easy to see that (except in one particular position) dy will be of a different size from dx.

Now right through the differential calculus we are hunting, hunting, hunting for a curious thing, a mere ratio, namely, the proportion which dy bears to dx when both of them are indefinitely small.

It should be noted here that we can only find this ratio $\frac{dy}{dx}$ when y and x are related to each other in some way, so that whenever x varies y does vary also. For instance, in the first example just taken, if the base x of the triangle be made longer, the height y of the triangle becomes greater also, and in the second example, if the distance x of the foot of the ladder from the wall be made to increase, the height y reached by the ladder decreases in a corresponding manner, slowly at first, but more and more rapidly as x becomes greater. In these cases the relation between x and y is perfectly definite, it can be expressed mathematically, being $\frac{y}{x} = \tan 30°$ and $x^2 + y^2 = l^2$ (where l is the length of the ladder) respectively, and $\frac{dy}{dx}$ has the meaning we found in each case.

If, while x is, as before, the distance of the foot of the ladder from the wall, y is, instead of the height reached, the horizontal length of the wall, or the number of bricks in it, or the number of years since it was built, any change in x would naturally cause no change whatever in y; in this case $\frac{dy}{dx}$ has no meaning whatever, and it is not possible to find an expression for it. Whenever we use differentials dx, dy, dz, etc., the existence of some kind of relation between x, y, z, etc., is implied, and this relation is called a "function" in x, y. z, etc.; the two expressions given above, for instance, namely $\frac{y}{x} = \tan 30^\circ$ and $x^2 + y^2 = l^2$, are functions of x and y. Such expressions contain implicitly (that is, contain without distinctly showing it) the means of expressing either x in terms of y or y in terms of x, and for this reason they are called *implicit functions* in x and y; they can be respectively put into the forms

$$y = x \tan 30^\circ \quad \text{or} \quad x = \frac{y}{\tan 30^\circ}$$

and $$y = \sqrt{l^2 - x^2} \quad \text{or} \quad x = \sqrt{l^2 - y^2}.$$

These last expressions state explicitly (that is, distinctly) the value of x in terms of y, or of y in terms of x, and they are for this reason called *explicit functions* of x or y. For example $x^2 + 3 = 2y - 7$ is an implicit function in x and y; it may be written $y = \frac{x^2 + 10}{2}$ (explicit function of x) or $x = \sqrt{2y - 10}$ (explicit function of y). We see that an explicit function in x, y, z, etc., is simply something the value of which changes when x, y, z, etc., are changing, either one at the time or several together. Because of this, the value of the explicit function is called the *dependent variable*, as it depends on the value of the other variable quantities in the function;

these other variables are called the *independent variables* because their value is not determined from the value assumed by the function. For example, if $u=x^2 \sin\theta$, x and θ are the independent variables, and u is the dependent variable.

Sometimes the exact relation between several quantities x, y, z either is not known or it is not convenient to state it; it is only known, or convenient to state, that there is some sort of relation between these variables, so that one cannot alter either x or y or z singly without affecting the other quantities; the existence of a function in x, y, z is then indicated by the notation $F(x, y, z)$ (implicit function) or by $x=F(y, z)$, $y=F(x, z)$ or $z=F(x, y)$ (explicit function). Sometimes the letter f or ϕ is used instead of F, so that $y=F(x)$, $y=f(x)$ and $y=\phi(x)$ all mean the same thing, namely, that the value of y depends on the value of x in some way which is not stated.

We call the ratio $\frac{dy}{dx}$, "the *differential coefficient* of y with respect to x." It is a solemn scientific name for this very simple thing. But we are not going to be frightened by solemn names, when the things themselves are so easy. Instead of being frightened we will simply pronounce a brief curse on the stupidity of giving long crack-jaw names; and, having relieved our minds, will go on to the simple thing itself, namely the ratio $\frac{dy}{dx}$.

In ordinary algebra which you learned at school, you were always hunting after some unknown quantity which you called x or y; or sometimes there were two unknown quantities to be hunted for simultaneously. You have now to learn to go hunting in a new way; the fox being now neither x nor y. Instead of this you have to hunt for this curious cub called $\frac{dy}{dx}$. The process of finding the value of $\frac{dy}{dx}$ is called "differentiating." But, remember, what is wanted is the value of this ratio when both dy and dx are themselves indefinitely small. The true value of the differential coefficient is that to which it approximates in the limiting case when each of them is considered as infinitesimally minute.

Let us now learn how to go in quest of $\frac{dy}{dx}$.

NOTE TO CHAPTER III.

How to read Differentials.

It will never do to fall into the schoolboy error of thinking that dx means d times x, for d is not a factor—it means "an element of" or "a bit of" whatever follows. One reads dx thus: "dee-eks."

In case the reader has no one to guide him in such matters it may here be simply said that one reads differential coefficients in the following way. The differential coefficient

$\frac{dy}{dx}$ is read "*dee-wy by dee-eks,*" or "*dee-wy over dee-eks.*"

So also $\frac{du}{dt}$ is read "*dee-you by dee-tee.*"

Second differential coefficients will be met with later on. They are like this:

$\frac{d^2y}{dx^2}$; which is read "*dee-two-wy over dee-eks-squared,*"

and it means that the operation of differentiating y with respect to x has been (or has to be) performed twice over.

Another way of indicating that a function has been differentiated is by putting an accent to the symbol of the function. Thus if $y=F(x)$, which means that y is some unspecified function of x (see p. 14), we may write $F'(x)$ instead of $\frac{d(F(x))}{dx}$. Similarly, $F''(x)$ will mean that the original function $F(x)$ has been differentiated twice over with respect to x.

CHAPTER IV.

SIMPLEST CASES.

Now let us see how, on first principles, we can differentiate some simple algebraical expression.

Case 1.

Let us begin with the simple expression $y=x^2$. Now remember that the fundamental notion about the calculus is the idea of *growing*. Mathematicians call it *varying*. Now as y and x^2 are equal to one another, it is clear that if x grows, x^2 will also grow. And if x^2 grows, then y will also grow. What we have got to find out is the proportion between the growing of y and the growing of x. In other words our task is to find out the ratio between dy and dx, or, in brief, to find the value of $\frac{dy}{dx}$.

Let x, then, grow a little bit bigger and become $x+dx$; similarly, y will grow a bit bigger and will become $y+dy$. Then, clearly, it will still be true that the enlarged y will be equal to the square of the enlarged x. Writing this down, we have:

$$y+dy=(x+dx)^2.$$

Doing the squaring we get:

$$y+dy=x^2+2x \cdot dx+(dx)^2.$$

What does $(dx)^2$ mean? Remember that dx meant a bit—a little bit—of x. Then $(dx)^2$ will mean a little bit of a little bit of x; that is, as explained above (p. 4), it is a small quantity of the second order of smallness. It may therefore be discarded as quite inconsiderable in comparison with the other terms Leaving it out, we then have:

$$y + dy = x^2 + 2x \cdot dx.$$

Now $y = x^2$; so let us subtract this from the equation and we have left

$$dy = 2x \cdot dx.$$

Dividing across by dx, we find

$$\frac{dy}{dx} = 2x.$$

Now *this** is what we set out to find. The ratio of the growing of y to the growing of x is, in the case before us, found to be $2x$.

* *N.B.*—This ratio $\frac{dy}{dx}$ is the result of differentiating y with respect to x. Differentiating means finding the differential coefficient. Suppose we had some other function of x, as, for example, $u = 7x^2 + 3$. Then if we were told to differentiate this with respect to x, we should have to find $\frac{du}{dx}$, or, what is the same thing, $\frac{d(7x^2+3)}{dx}$. On the other hand, we may have a case in which time was the independent variable (see p. 15), such as this: $y = b + \frac{1}{2}at^2$. Then, if we were told to differentiate it, that means we must find its differential coefficient with respect to t. So that then our business would be to try to find $\frac{dy}{dt}$, that is, to find $\frac{d(b+\frac{1}{2}at^2)}{dt}$.

Numerical example.

Suppose $x = 100$ and $\therefore\ y = 10{,}000$. Then let x grow till it becomes 101 (that is, let $dx = 1$). Then the enlarged y will be $101 \times 101 = 10{,}201$. But if we agree that we may ignore small quantities of the second order, 1 may be rejected as compared with 10,000; so we may round off the enlarged y to 10,200. y has grown from 10,000 to 10,200; the bit added on is dy, which is therefore 200.

$\dfrac{dy}{dx} = \dfrac{200}{1} = 200$. According to the algebra-working of the previous paragraph, we find $\dfrac{dy}{dx} = 2x$. And so it is; for $x = 100$ and $2x = 200$.

But, you will say, we neglected a whole unit.

Well, try again, making dx a still smaller bit.

Try $dx = \frac{1}{10}$. Then $x + dx = 100{\cdot}1$, and

$$(x + dx)^2 = 100{\cdot}1 \times 100{\cdot}1 = 10{,}020{\cdot}01.$$

Now the last figure 1 is only one-millionth part of the 10,000, and is utterly negligible; so we may take 10,020 without the little decimal at the end. And this makes $dy = 20$; and $\dfrac{dy}{dx} = \dfrac{20}{0{\cdot}1} = 200$, which is still the same as $2x$.

Case 2.

Try differentiating $y = x^3$ in the same way.

We let y grow to $y + dy$, while x grows to $x + dx$. Then we have

$$y + dy = (x + dx)^3.$$

Doing the cubing we obtain

$$y+dy=x^3+3x^2\cdot dx+3x(dx)^2+(dx)^3.$$

Now we know that we may neglect small quantities of the second and third orders; since, when dy and dx are both made indefinitely small, $(dx)^2$ and $(dx)^3$ will become indefinitely smaller by comparison. So, regarding them as negligible, we have left:

$$y+dy=x^3+3x^2\cdot dx.$$

But $y=x^3$; and, subtracting this, we have:

$$dy=3x^2\cdot dx,$$

and

$$\frac{dy}{dx}=3x^2.$$

Case 3.

Try differentiating $y=x^4$. Starting as before by letting both y and x grow a bit, we have:

$$y+dy=(x+dx)^4.$$

Working out the raising to the fourth power, we get

$$y+dy=x^4+4x^3dx+6x^2(dx)^2+4x(dx)^3+(dx)^4.$$

Then, striking out the terms containing all the higher powers of dx, as being negligible by comparison, we have

$$y+dy=x^4+4x^3dx.$$

Subtracting the original $y=x^4$, we have left

$$dy=4x^3dx,$$

and

$$\frac{dy}{dx}=4x^3.$$

Now all these cases are quite easy. Let us collect the results to see if we can infer any general rule. Put them in two columns, the values of y in one and the corresponding values found for $\frac{dy}{dx}$ in the other: thus

y	$\frac{dy}{dx}$
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

Just look at these results: the operation of differentiating appears to have had the effect of diminishing the power of x by 1 (for example in the last case reducing x^4 to x^3), and at the same time multiplying by a number (the same number in fact which originally appeared as the power). Now, when you have once seen this, you might easily conjecture how the others will run. You would expect that differentiating x^5 would give $5x^4$, or differentiating x^6 would give $6x^5$. If you hesitate, try one of these, and see whether the conjecture comes right.

Try $y=x^5$.

Then $y+dy=(x+dx)^5$

$$=x^5+5x^4dx+10x^3(dx)^2+10x^2(dx)^3+5x(dx)^4+(dx)^5.$$

Neglecting all the terms containing small quantities of the higher orders, we have left

$$y+dy=x^5+5x^4dx,$$

and subtracting $y=x^5$ leaves us

$$dy = 5x^4 dx,$$

whence $$\frac{dy}{dx}=5x^4,$$ exactly as we supposed.

Following out logically our observation, we should conclude that if we want to deal with any higher power,—call it n—we could tackle it in the same way.

Let $$y=x^n,$$

then, we should expect to find that

$$\frac{dy}{dx}=nx^{(n-1)}.$$

For example, let $n=8$, then $y=x^8$; and differentiating it would give $\frac{dy}{dx}=8x^7$.

And, indeed, the rule that differentiating x^n gives as the result nx^{n-1} is true for all cases where n is a whole number and positive. [Expanding $(x+dx)^n$ by the binomial theorem will at once show this.] But the question whether it is true for cases where n has negative or fractional values requires further consideration.

Case of a negative power.

Let $y=x^{-2}$. Then proceed as before:

$$\begin{aligned} y+dy &= (x+dx)^{-2} \\ &= x^{-2}\left(1+\frac{dx}{x}\right)^{-2}. \end{aligned}$$

Expanding this by the binomial theorem (see p. 141), we get

$$=x^{-2}\left[1-\frac{2\,dx}{x}+\frac{2(2+1)}{1\times 2}\left(\frac{dx}{x}\right)^2-\text{etc.}\right]$$

$$=x^{-2}-2x^{-3}\cdot dx+3x^{-4}(dx)^2-4x^{-5}(dx)^3+\text{etc.}$$

So, neglecting the small quantities of higher orders of smallness, we have:

$$y+dy=x^{-2}-2x^{-3}\cdot dx.$$

Subtracting the original $y=x^{-2}$, we find

$$dy=-2x^{-3}dx,$$

$$\frac{dy}{dx}=-2x^{-3}.$$

And this is still in accordance with the rule inferred above.

Case of a fractional power.

Let $y=x^{\frac{1}{2}}$. Then, as before,

$$y+dy=(x+dx)^{\frac{1}{2}}=x^{\frac{1}{2}}\left(1+\frac{dx}{x}\right)^{\frac{1}{2}}$$

$$=\sqrt{x}+\frac{1}{2}\frac{dx}{\sqrt{x}}-\frac{1}{8}\frac{(dx)^2}{x\sqrt{x}}+\text{terms with higher powers of } dx.$$

Subtracting the original $y=x^{\frac{1}{2}}$, and neglecting higher powers we have left:

$$dy=\frac{1}{2}\frac{dx}{\sqrt{x}}=\frac{1}{2}x^{-\frac{1}{2}}\cdot dx,$$

and $\dfrac{dy}{dx}=\dfrac{1}{2}x^{-\frac{1}{2}}$. This agrees with the general rule.

Summary. Let us see how far we have got. We have arrived at the following rule: To differentiate x^n, multiply it by the power and reduce the power by one, so giving us nx^{n-1} as the result.

Exercises I. (See p. 288 for Answers.)

Differentiate the following:

(1) $y = x^{13}$ (2) $y = x^{-\frac{3}{2}}$

(3) $y = x^{2a}$ (4) $u = t^{2.4}$

(5) $z = \sqrt[3]{u}$ (6) $y = \sqrt[3]{x^{-5}}$

(7) $u = \sqrt[5]{\dfrac{1}{x^8}}$ (8) $y = 2x^a$.

(9) $y = \sqrt[q]{x^3}$ (10) $y = \sqrt[n]{\dfrac{1}{x^m}}$

You have now learned how to differentiate powers of x. How easy it is!

CHAPTER V.

NEXT STAGE. WHAT TO DO WITH CONSTANTS.

IN our equations we have regarded x as growing, and as a result of x being made to grow y also changed its value and grew. We usually think of x as a quantity that we can vary; and, regarding the variation of x as a sort of *cause*, we consider the resulting variation of y as an *effect*. In other words, we regard the value of y as depending on that of x. Both x and y are variables, but x is the one that we operate upon, and y is the "dependent variable." In all the preceding chapter we have been trying to find out rules for the proportion which the dependent variation in y bears to the variation independently made in x.

Our next step is to find out what effect on the process of differentiating is caused by the presence of *constants*, that is, of numbers which don't change when x or y changes its value.

Added Constants.

Let us begin with some simple case of an added constant, thus:

Let $$y = x^3 + 5.$$

Just as before, let us suppose x to grow to $x + dx$ and y to grow to $y + dy$.

Then: $$y+dy=(x+dx)^3+5$$
$$=x^3+3x^2dx+3x(dx)^2+(dx)^3+5.$$

Neglecting the small quantities of higher orders, this becomes $$y+dy=x^3+3x^2\cdot dx+5.$$

Subtract the original $y=x^3+5$, and we have left:

$$dy=3x^2dx.$$

$$\frac{dy}{dx}=3x^2.$$

So the 5 has quite disappeared. It added nothing to the growth of x, and does not enter into the differential coefficient. If we had put 7, or 700, or any other number, instead of 5, it would have disappeared. So if we take the letter a, or b, or c to represent any constant, it will simply disappear when we differentiate.

If the additional constant had been of negative value, such as -5 or $-b$, it would equally have disappeared.

Multiplied Constants.

Take as a simple experiment this case:

Let $y=7x^2$.

Then on proceding as before we get:

$$y+dy=7(x+dx)^2$$
$$=7\{x^2+2x\cdot dx+(dx)^2\}$$
$$=7x^2+14x\cdot dx+7(dx)^2.$$

Then, subtracting the original $y=7x^2$, and neglecting the last term, we have

$$dy=14x\cdot dx.$$

$$\frac{dy}{dx}=14x.$$

Let us illustrate this example by working out the graphs of the equations $y=7x^2$ and $\frac{dy}{dx}=14x$, by assigning to x a set of successive values, 0, 1, 2, 3, etc., and finding the corresponding values of y and of $\frac{dy}{dx}$.

These values we tabulate as follows:

x	0	1	2	3	4	5	−1	−2	−3
y	0	7	28	63	112	175	7	28	63
$\frac{dy}{dx}$	0	14	28	42	56	70	−14	−28	−42

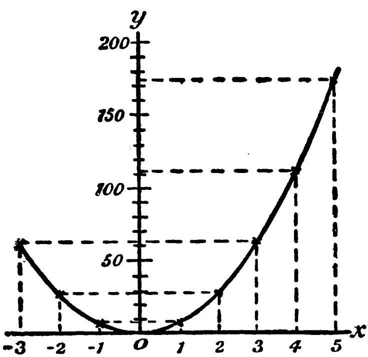

FIG. 6.—Graph of $y=7x^2$.

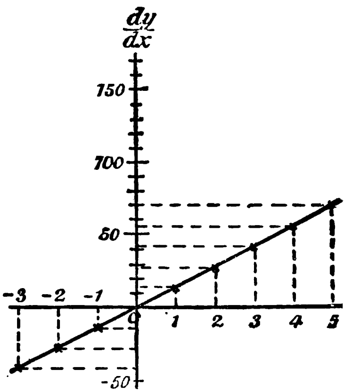

FIG. 6a.—Graph of $\frac{dy}{dx}=14x$.

Now plot these values to some convenient scale, and we obtain the two curves, Figs. 6 and 6*a*.

Carefully compare the two figures, and verify by inspection that the height of the ordinate of the derived curve, Fig. 6*a*, is proportional to the *slope* of the original curve,* Fig. 6, at the corresponding value of x. To the left of the origin, where the original curve slopes negatively (that is, downward from left to right) the corresponding ordinates of the derived curve are negative.

Now, if we look back at p. 19, we shall see that simply differentiating x^2 gives us $2x$. So that the differential coefficient of $7x^2$ is just 7 times as big as that of x^2. If we had taken $8x^2$, the differential coefficient would have come out eight times as great as that of x^2. If we put $y=ax^2$, we shall get

$$\frac{dy}{dx}=a\times 2x.$$

If we had begun with $y=ax^n$, we should have had $\frac{dy}{dx}=a\times nx^{n-1}$. So that any mere multiplication by a constant reappears as a mere multiplication when the thing is differentiated. And, what is true about multiplication is equally true about *division*: for if, in the example above, we had taken as the constant $\frac{1}{7}$ instead of 7, we should have had the same $\frac{1}{7}$ come out in the result after differentiation.

* See p. 77 about *slopes* of curves.

Some Further Examples.

The following further examples, fully worked out, will enable you to master completely the process of differentiation as applied to ordinary algebraical expressions, and enable you to work out by yourself the examples given at the end of this chapter.

(1) Differentiate $y=\frac{x^5}{7}-\frac{3}{5}$.

$\frac{3}{5}$ is an added constant and vanishes (see p. 26).

We may then write at once

$$\frac{dy}{dx}=\frac{1}{7}\times 5\times x^{5-1},$$

or

$$\frac{dy}{dx}=\frac{5}{7}x^4.$$

(2) Differentiate $y=a\sqrt{x}-\frac{1}{2}\sqrt{a}$.

The term $\frac{1}{2}\sqrt{a}$ vanishes, being an added constant; and as $a\sqrt{x}$, in the index form, is written $ax^{\frac{1}{2}}$, we have

$$\frac{dy}{dx}=a\times\frac{1}{2}\times x^{\frac{1}{2}-1}=\frac{a}{2}\times x^{-\frac{1}{2}},$$

or

$$\frac{dy}{dx}=\frac{a}{2\sqrt{x}}.$$

(3) If $ay+bx=by-ax+(x+y)\sqrt{a^2-b^2}$,

find the differential coefficient of y with respect to x.

As a rule an expression of this kind will need a little more knowledge than we have acquired so far;

it is, however, always worth while to try whether the expression can be put in a simpler form.

First we must try to bring it into the form $y=$ some expression involving x only.

The expression may be written

$$(a-b)y+(a+b)x=(x+y)\sqrt{a^2-b^2}.$$

Squaring, we get

$$(a-b)^2y^2+(a+b)^2x^2+2(a+b)(a-b)xy$$
$$=(x^2+y^2+2xy)(a^2-b^2),$$

which simplifies to

$$(a-b)^2y^2+(a+b)^2x^2=x^2(a^2-b^2)+y^2(a^2-b^2);$$

or $\quad [(a-b)^2-(a^2-b^2)]y^2=[(a^2-b^2)-(a+b)^2]x^2,$

that is $\quad 2b(b-a)y^2=-2b(b+a)x^2;$

hence $\quad y=\sqrt{\dfrac{a+b}{a-b}}\,x \quad$ and $\quad \dfrac{dy}{dx}=\sqrt{\dfrac{a+b}{a-b}}.$

(4) The volume of a cylinder of radius r and height h is given by the formula $V=\pi r^2 h$. Find the rate of variation of volume with the radius when $r=5{\cdot}5$ in. and $h=20$ in. If $r=h$, find the dimensions of the cylinder so that a change of 1 in. in radius causes a change of 400 cub. in. in the volume.

The rate of variation of V with regard to r is

$$\frac{dV}{dr}=2\pi rh.$$

If $r=5{\cdot}5$ in. and $h=20$ in. this becomes 690·8. It means that a change of radius of 1 inch will cause a change of volume of 690·8 cub. inch. This can be easily verified, for the volumes with $r=5$ and $r=6$

are 1570 cub. in. and 2260·8 cub. in. respectively, and $2260{\cdot}8-1570=690{\cdot}8$.

Also, if

$$r=h,\quad \frac{dV}{dr}=2\pi r^2=400 \quad\text{and}\quad r=h=\sqrt{\frac{400}{2\pi}}=7{\cdot}98 \text{ in.}$$

(5) The reading θ of a Féry's Radiation pyrometer is related to the Centigrade temperature t of the observed body by the relation

$$\frac{\theta}{\theta_1}=\left(\frac{t}{t_1}\right)^4,$$

where θ_1 is the reading corresponding to a known temperature t_1 of the observed body.

Compare the sensitiveness of the pyrometer at temperatures 800° C., 1000° C., 1200° C., given that it read 25 when the temperature was 1000° C.

The sensitiveness is the rate of variation of the reading with the temperature, that is $\dfrac{d\theta}{dt}$. The formula may be written

$$\theta=\frac{\theta_1}{t_1^{\,4}}\,t^4=\frac{25t^4}{1000^4},$$

and we have

$$\frac{d\theta}{dt}=\frac{100t^3}{1000^4}=\frac{t^3}{10{,}000{,}000{,}000}.$$

When $t=800$, 1000 and 1200, we get $\dfrac{d\theta}{dt}=0{\cdot}0512$, $0{\cdot}1$ and $0{\cdot}1728$ respectively.

The sensitiveness is approximately doubled from 800° to 1000°, and becomes three-quarters as great again up to 1200°

Exercises II. (See p. 288 for Answers.)

Differentiate the following:

(1) $y = ax^3 + 6$. (2) $y = 13x^{\frac{3}{2}} - c$.

(3) $y = 12x^{\frac{1}{2}} + c^{\frac{1}{2}}$. (4) $y = c^{\frac{1}{2}} x^{\frac{1}{2}}$.

(5) $u = \dfrac{az^n - 1}{c}$. (6) $y = 1{\cdot}18t^2 + 22{\cdot}4$.

Make up some other examples for yourself, and try your hand at differentiating them.

(7) If l_t and l_0 be the lengths of a rod of iron at the temperatures $t°$ C. and $0°$ C. respectively, then $l_t = l_0(1 + 0{\cdot}000012t)$. Find the change of length of the rod per degree Centigrade.

(8) It has been found that if c be the candle power of an incandescent electric lamp, and V be the voltage, $c = aV^b$, where a and b are constants.

Find the rate of change of the candle power with the voltage, and calculate the change of candle power per volt at 80, 100 and 120 volts in the case of a lamp for which $a = 0{\cdot}5 \times 10^{-10}$ and $b = 6$.

(9) The frequency n of vibration of a string of diameter D, length L and specific gravity σ, stretched with a force T, is given by

$$n = \frac{1}{DL}\sqrt{\frac{gT}{\pi\sigma}}.$$

Find the rate of change of the frequency when D, L, σ and T are varied singly.

(10) The greatest external pressure P which a tube can support without collapsing is given by

$$P=\left(\frac{2E}{1-\sigma^2}\right)\frac{t^3}{D^3},$$

where E and σ are constants, t is the thickness of the tube and D is its diameter. (This formula assumes that $4t$ is small compared to D.)

Compare the rate at which P varies for a small change of thickness and for a small change of diameter taking place separately.

(11) Find, from first principles, the rate at which the following vary with respect to a change in radius:

(*a*) the circumference of a circle of radius r;
(*b*) the area of a circle of radius r;
(*c*) the lateral area of a cone of slant dimension l;
(*d*) the volume of a cone of radius r and height h;
(*e*) the area of a sphere of radius r;
(*f*) the volume of a sphere of radius r.

(12) The length L of an iron rod at the temperature T being given by $L=l_t[1+0{\cdot}000012(T-t)]$, where l_t is the length at the temperature t, find the rate of variation of the diameter D of an iron tyre suitable for being shrunk on a wheel, when the temperature T varies.

CHAPTER VI.

SUMS, DIFFERENCES, PRODUCTS, AND QUOTIENTS.

We have learned how to differentiate simple algebraical functions such as x^2+c or ax^4, and we have now to consider how to tackle the *sum* of two or more functions.

For instance, let

$$y=(x^2+c)+(ax^4+b);$$

what will its $\frac{dy}{dx}$ be? How are we to go to work on this new job?

The answer to this question is quite simple: just differentiate them, one after the other, thus:

$$\frac{dy}{dx}=2x+4ax^3. \quad (Ans.)$$

If you have any doubt whether this is right, try a more general case, working it by first principles. And this is the way.

Let $y=u+v$, where u is any function of x, and v any other function of x. Then, letting x increase to $x+dx$, y will increase to $y+dy$; and u will increase to $u+du$; and v to $v+dv$.

And we shall have:

$$y+dy=u+du+v+dv.$$

Subtracting the original $y=u+v$, we get

$$dy=du+dv,$$

and dividing through by dx, we get:

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}.$$

This justifies the procedure. You differentiate each function separately and add the results. So if now we take the example of the preceding paragraph, and put in the values of the two functions, we shall have, using the notation shown (p. 17),

$$\frac{dy}{dx}=\frac{d(x^2+c)}{dx}+\frac{d(ax^4+b)}{dx}$$
$$=2x \qquad +4ax^3,$$

exactly as before.

If there were three functions of x, which we may call u, v and w, so that

$$y=u+v+w;$$

then $$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}.$$

As for the rule about *subtraction*, it follows at once; for if the function v had itself had a negative sign, its differential coefficient would also be negative; so that by differentiating

$$y=u-v,$$

we should get $$\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}.$$

But when we come to do with *Products*, the thing is not quite so simple.

Suppose we were asked to differentiate the expression

$$y=(x^2+c)\times(ax^4+b),$$

what are we to do? The result will certainly *not* be $2x\times4ax^3$; for it is easy to see that neither $c\times ax^4$, nor $x^2\times b$, would have been taken into that product.

Now there are two ways in which we may go to work.

First way. Do the multiplying first, and, having worked it out, then differentiate.

Accordingly, we multiply together x^2+c and ax^4+b.

This gives $ax^6+acx^4+bx^2+bc$.

Now differentiate, and we get:

$$\frac{dy}{dx}=6ax^5+4acx^3+2bx.$$

Second way. Go back to first principles, and consider the equation

$$y=u\times v;$$

where u is one function of x, and v is any other function of x. Then, if x grows to be $x+dx$; and y to $y+dy$; and u becomes $u+du$; and v becomes $v+dv$, we shall have:

$$\begin{aligned} y+dy&=(u+du)\times(v+dv)\\ &=u\cdot v+u\cdot dv+v\cdot du+du\cdot dv. \end{aligned}$$

Now $du\cdot dv$ is a small quantity of the second order of smallness, and therefore in the limit may be discarded, leaving

$$y+dy=u\cdot v+u\cdot dv+v\cdot du.$$

Then, subtracting the original $y=u\cdot v$, we have left

$$dy=u\cdot dv+v\cdot du;$$

and, dividing through by dx, we get the result:

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

This shows that our instructions will be as follows: *To differentiate the product of two functions, multiply each function by the differential coefficient of the other, and add together the two products so obtained.*

You should note that this process amounts to the following: Treat u as constant while you differentiate v; then treat v as constant while you differentiate u; and the whole differential coefficient $\frac{dy}{dx}$ will be the sum of the results of these two treatments.

Now, having found this rule, apply it to the concrete example which was considered above.

We want to differentiate the product

$$(x^2+c)\times(ax^4+b).$$

Call $(x^2+c)=u$; and $(ax^4+b)=v$.

Then, by the general rule just established, we may write:

$$\begin{aligned}\frac{dy}{dx}&=(x^2+c)\frac{d(ax^4+b)}{dx}+(ax^4+b)\frac{d(x^2+c)}{dx}\\&=(x^2+c)4ax^3 \qquad +(ax^4+b)2x\\&=4ax^5+4acx^3 \qquad +2ax^5+2bx,\end{aligned}$$

$$\frac{dy}{dx}=6ax^5+4acx^3 \qquad +2bx,$$

exactly as before.

Lastly, we have to differentiate *quotients.*

Think of this example, $y=\frac{bx^5+c}{x^2+a}$. In such a case it is no use to try to work out the division beforehand, because x^2+a will not divide into bx^5+c, neither have they any common factor. So there is nothing for it but to go back to first principles, and find a rule.

So we will put $$y=\frac{u}{v};$$
where u and v are two different functions of the independent variable x. Then, when x becomes $x+dx$, y will become $y+dy$; and u will become $u+du$; and v will become $v+dv$. So then

$$y+dy=\frac{u+du}{v+dv}.$$

Now perform the algebraic division, thus:

$$\begin{array}{r|l|l}
v+dv & u+du & \frac{u}{v}+\frac{du}{v}-\frac{u\cdot dv}{v^2} \\
 & u+\frac{u\cdot dv}{v} & \\
\hline
 & \quad du-\frac{u\cdot dv}{v} & \\
 & \quad du+\frac{du\cdot dv}{v} & \\
\hline
 & \qquad -\frac{u\cdot dv}{v}-\frac{du\cdot dv}{v} & \\
 & \qquad -\frac{u\cdot dv}{v}-\frac{u\cdot dv\cdot dv}{v^2} & \\
\hline
 & \qquad\qquad -\frac{du\cdot dv}{v}+\frac{u\cdot dv\cdot dv}{v^2}. &
\end{array}$$

As both these remainders are small quantities of the second order, they may be neglected, and the division may stop here, since any further remainders would be of still smaller magnitudes.

So we have got:

$$y+dy=\frac{u}{v}+\frac{du}{v}-\frac{u\cdot dv}{v^2};$$

which may be written

$$=\frac{u}{v}+\frac{v\cdot du-u\cdot dv}{v^2}.$$

Now subtract the original $y=\frac{u}{v}$, and we have left:

$$dy=\frac{v\cdot du-u\cdot dv}{v^2};$$

whence

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

This gives us our instructions as to *how to differentiate a quotient* of two functions. *Multiply the divisor function by the differential coefficient of the dividend function; then multiply the dividend function by the differential coefficient of the divisor function; and subtract the latter product from the former. Lastly, divide the difference by the square of the divisor function.*

Going back to our example $y=\frac{bx^5+c}{x^2+a}$,

write $bx^5+c=u$;

and $x^2+a=v$.

Then

$$\frac{dy}{dx}=\frac{(x^2+a)\dfrac{d(bx^5+c)}{dx}-(bx^5+c)\dfrac{d(x^2+a)}{dx}}{(x^2+a)^2}$$

$$=\frac{(x^2+a)(5bx^4)-(bx^5+c)(2x)}{(x^2+a)^2},$$

$$\frac{dy}{dx}=\frac{3bx^6+5abx^4-2cx}{(x^2+a)^2}. \quad (\textit{Answer.})$$

The working out of quotients is often tedious, but there is nothing difficult about it.

Some further examples fully worked out are given hereafter.

(1) Differentiate $y=\frac{a}{b^2}x^3-\frac{a^2}{b}x+\frac{a^2}{b^2}$.

Being a constant, $\frac{a^2}{b^2}$ vanishes, and we have

$$\frac{dy}{dx}=\frac{a}{b^2}\times 3\times x^{3-1}-\frac{a^2}{b}\times 1\times x^{1-1}.$$

But $x^{1-1}=x^0=1$; so we get:

$$\frac{dy}{dx}=\frac{3a}{b^2}x^2-\frac{a^2}{b}.$$

(2) Differentiate $y=2a\sqrt{bx^3}-\frac{3b\sqrt[3]{a}}{x}-2\sqrt{ab}$.

Putting x in the index form, we get

$$y=2a\sqrt{b}\,x^{\frac{3}{2}}-3b\sqrt[3]{a}\,x^{-1}-2\sqrt{ab}.$$

Now

$$\frac{dy}{dx}=2a\sqrt{b}\times\tfrac{3}{2}\times x^{\frac{3}{2}-1}-3b\sqrt[3]{a}\times(-1)\times x^{-1-1};$$

or,

$$\frac{dy}{dx}=3a\sqrt{bx}+\frac{3b\sqrt[3]{a}}{x^2}.$$

(3) Differentiate $z=1{\cdot}8\sqrt[3]{\frac{1}{\theta^2}}-\frac{4{\cdot}4}{\sqrt[5]{\theta}}-27°.$

This may be written: $z=1{\cdot}8\,\theta^{-\frac{2}{3}}-4{\cdot}4\,\theta^{-\frac{1}{5}}-27°.$

The 27° vanishes, and we have

$$\frac{dz}{d\theta}=1{\cdot}8\times-\tfrac{2}{3}\times\theta^{-\frac{2}{3}-1}-4{\cdot}4\times(-\tfrac{1}{5})\theta^{-\frac{1}{5}-1};$$

or,
$$\frac{dz}{d\theta}=-1{\cdot}2\,\theta^{-\frac{5}{3}}+0{\cdot}88\,\theta^{-\frac{6}{5}};$$

or,
$$\frac{dz}{d\theta}=\frac{0{\cdot}88}{\sqrt[5]{\theta^6}}-\frac{1{\cdot}2}{\sqrt[3]{\theta^5}}.$$

(4) Differentiate $v=(3t^2-1{\cdot}2t+1)^3.$

A direct way of doing this will be explained later (see p. 67); but we can nevertheless manage it now without any difficulty.

Developing the cube, we get

$$v=27t^6-32{\cdot}4t^5+39{\cdot}96t^4-23{\cdot}328t^3+13{\cdot}32t^2-3{\cdot}6t+1;$$

hence

$$\frac{dv}{dt}=162t^5-162t^4+159{\cdot}84t^3-69{\cdot}984t^2+26{\cdot}64t-3{\cdot}6.$$

(5) Differentiate $y=(2x-3)(x+1)^2.$

$$\frac{dy}{dx}=(2x-3)\frac{d[(x+1)(x+1)]}{dx}+(x+1)^2\frac{d(2x-3)}{dx}$$

$$=(2x-3)\left[(x+1)\frac{d(x+1)}{dx}+(x+1)\frac{d(x+1)}{dx}\right]+(x+1)^2\frac{d(2x-3)}{dx}$$

$$=2(x+1)[(2x-3)+(x+1)]=2(x+1)(3x-2);$$

or, more simply, multiply out and then differentiate.

(6) Differentiate $y=0{\cdot}5x^3(x-3)$.

$$\frac{dy}{dx}=0{\cdot}5\left[x^3\frac{d(x-3)}{dx}+(x-3)\frac{d(x^3)}{dx}\right]$$

$$=0{\cdot}5[x^3+(x-3)\times 3x^2]=2x^3-4{\cdot}5x^2.$$

Same remarks as for preceding example.

(7) Differentiate $w=\left(\theta+\frac{1}{\theta}\right)\left(\sqrt{\theta}+\frac{1}{\sqrt{\theta}}\right)$.

This may be written

$$w=(\theta+\theta^{-1})(\theta^{\frac{1}{2}}+\theta^{-\frac{1}{2}}).$$

$$\frac{dw}{d\theta}=(\theta+\theta^{-1})\frac{d(\theta^{\frac{1}{2}}+\theta^{-\frac{1}{2}})}{d\theta}+(\theta^{\frac{1}{2}}+\theta^{-\frac{1}{2}})\frac{d(\theta+\theta^{-1})}{d\theta}$$

$$=(\theta+\theta^{-1})(\tfrac{1}{2}\theta^{-\frac{1}{2}}-\tfrac{1}{2}\theta^{-\frac{3}{2}})+(\theta^{\frac{1}{2}}+\theta^{-\frac{1}{2}})(1-\theta^{-2})$$

$$=\tfrac{1}{2}(\theta^{\frac{1}{2}}+\theta^{-\frac{3}{2}}-\theta^{-\frac{1}{2}}-\theta^{-\frac{5}{2}})+(\theta^{\frac{1}{2}}+\theta^{-\frac{1}{2}}-\theta^{-\frac{3}{2}}-\theta^{-\frac{5}{2}})$$

$$=\tfrac{3}{2}\left(\sqrt{\theta}-\frac{1}{\sqrt{\theta^5}}\right)+\tfrac{1}{2}\left(\frac{1}{\sqrt{\theta}}-\frac{1}{\sqrt{\theta^3}}\right).$$

This, again, could be obtained more simply by multiplying the two factors first, and differentiating afterwards. This is not, however, always possible; see, for instance, p. 173, example 8, in which the rule for differentiating a product *must* be used.

(8) Differentiate $y=\dfrac{a}{1+a\sqrt{x}+a^2x}$.

$$\frac{dy}{dx}=\frac{(1+ax^{\frac{1}{2}}+a^2x)\times 0-a\dfrac{d(1+ax^{\frac{1}{2}}+a^2x)}{dx}}{(1+a\sqrt{x}+a^2x)^2}$$

$$=-\frac{a(\frac{1}{2}ax^{-\frac{1}{2}}+a^2)}{(1+ax^{\frac{1}{2}}+a^2x)^2}.$$

(9) Differentiate $y=\dfrac{x^2}{x^2+1}$.

$$\frac{dy}{dx}=\frac{(x^2+1)2x-x^2\times 2x}{(x^2+1)^2}=\frac{2x}{(x^2+1)^2}.$$

(10) Differentiate $y=\dfrac{a+\sqrt{x}}{a-\sqrt{x}}$.

In the indexed form, $y=\dfrac{a+x^{\frac{1}{2}}}{a-x^{\frac{1}{2}}}$.

$$\frac{dy}{dx}=\frac{(a-x^{\frac{1}{2}})(\frac{1}{2}x^{-\frac{1}{2}})-(a+x^{\frac{1}{2}})(-\frac{1}{2}x^{-\frac{1}{2}})}{(a-x^{\frac{1}{2}})^2}=\frac{a-x^{\frac{1}{2}}+a+x^{\frac{1}{2}}}{2(a-x^{\frac{1}{2}})^2x^{\frac{1}{2}}};$$

hence $$\frac{dy}{dx}=\frac{a}{(a-\sqrt{x})^2\sqrt{x}}.$$

(11) Differentiate $\theta=\dfrac{1-a\sqrt[3]{t^2}}{1+a\sqrt[2]{t^3}}$.

Now $$\theta=\frac{1-at^{\frac{2}{3}}}{1+at^{\frac{3}{2}}}.$$

$$\frac{d\theta}{dt}=\frac{(1+at^{\frac{3}{2}})(-\frac{2}{3}at^{-\frac{1}{3}})-(1-at^{\frac{2}{3}})\times\frac{3}{2}at^{\frac{1}{2}}}{(1+at^{\frac{3}{2}})^2}$$

$$=\frac{5a^2\sqrt[6]{t^7}-\dfrac{4a}{\sqrt[3]{t}}-9a\sqrt[2]{t}}{6(1+a\sqrt[2]{t^3})^2}.$$

(12) A reservoir of square cross-section has sides sloping at an angle of 45° with the vertical. The side

of the bottom is 200 feet. Find an expression for the quantity pouring in or out when the depth of water varies by 1 foot; hence find, in gallons, the quantity withdrawn hourly when the depth is reduced from 14 to 10 feet in 24 hours.

The volume of a frustum of pyramid of height H, and of bases A and a, is $V=\frac{H}{3}(A+a+\sqrt{Aa})$. It is easily seen that, the slope being 45°, if the depth be h, the length of the side of the square surface of the water is $200+2h$ feet, so that the volume of water is

$$\frac{h}{3}[200^2+(200+2h)^2+200(200+2h)]$$
$$=40{,}000h+400h^2+\frac{4h^3}{3}.$$

$\frac{dV}{dh}=40{,}000+800h+4h^2=$ cubic feet per foot of depth variation. The mean level from 14 to 10 feet is 12 feet, when $h=12$, $\frac{dV}{dh}=50{,}176$ cubic feet.

Gallons per hour corresponding to a change of depth of 4 ft. in 24 hours $=\frac{4\times 50{,}176\times 6{\cdot}25}{24}=52{,}267$ gallons.

(13) The absolute pressure, in atmospheres, P, of saturated steam at the temperature $t°$ C. is given by Dulong as being $P=\left(\frac{40+t}{140}\right)^5$ as long as t is above 80°. Find the rate of variation of the pressure with the temperature at 100° C.

Expand the numerator by the binomial theorem (see p. 141).

$$P=\frac{1}{140^5}(40^5+5\times 40^4t+10\times 40^3t^2+10\times 40^2t^3 + 5\times 40t^4+t^5);$$

hence $$\frac{dP}{dt}=\frac{1}{537{,}824\times 10^5}$$

$$(5\times 40^4+20\times 40^3t+30\times 40^2t^2+20\times 40t^3+5t^4),$$

when $t=100$ this becomes 0·036 atmosphere per degree Centigrade change of temperature.

Exercises III. (See the Answers on p. 289.)

(1) Differentiate

(*a*) $u=1+x+\frac{x^2}{1\times 2}+\frac{x^3}{1\times 2\times 3}+\ldots.$

(*b*) $y=ax^2+bx+c.$ (*c*) $y=(x+a)^2.$

(*d*) $y=(x+a)^3.$

(2) If $w=at-\frac{1}{2}bt^2$, find $\frac{dw}{dt}$.

(3) Find the differential coefficient of

$$y=(x+\sqrt{-1})\times(x-\sqrt{-1}).$$

(4) Differentiate

$$y=(197x-34x^2)\times(7+22x-83x^3).$$

(5) If $x=(y+3)\times(y+5)$, find $\frac{dx}{dy}$.

(6) Differentiate $y=1·3709x\times(112·6+45·202x^2)$.

Find the differential coefficients of

(7) $y=\frac{2x+3}{3x+2}$. (8) $y=\frac{1+x+2x^2+3x^3}{1+x+2x^2}$.

(9) $y=\frac{ax+b}{cx+d}$. (10) $y=\frac{x^n+a}{x^{-n}+b}$.

(11) The temperature t of the filament of an incandescent electric lamp is connected to the current passing through the lamp by the relation

$$C=a+bt+ct^2.$$

Find an expression giving the variation of the current corresponding to a variation of temperature.

(12) The following formulae have been proposed to express the relation between the electric resistance R of a wire at the temperature t° C., and the resistance R_0 of that same wire at 0° Centigrade, a and b being constants.

$$R=R_0(1+at+bt^2).$$
$$R=R_0(1+at+b\sqrt{t}).$$
$$R=R_0(1+at+bt^2)^{-1}.$$

Find the rate of variation of the resistance with regard to temperature as given by each of these formulae.

(13) The electromotive-force E of a certain type of standard cell has been found to vary with the temperature t according to the relation

$$E=1{\cdot}4340[1-0{\cdot}000814(t-15) + 0{\cdot}000007(t-15)^2] \text{ volts.}$$

Find the change of electromotive-force per degree, at 15°, 20° and 25°.

(14) The electromotive-force necessary to maintain an electric arc of length l with a current of intensity i has been found by Mrs. Ayrton to be

$$E = a + bl + \frac{c + kl}{i},$$

where a, b, c, k are constants.

Find an expression for the variation of the electromotive force (*a*) with regard to the length of the arc; (*b*) with regard to the strength of the current.

CHAPTER VII.

SUCCESSIVE DIFFERENTIATION.

LET us try the effect of repeating several times over the operation of differentiating a function (see p. 14). Begin with a concrete case.

Let $y=x^5$.

First differentiation,	$5x^4$.	
Second differentiation,	$5\times 4x^3$	$=20x^3$.
Third differentiation,	$5\times 4\times 3x^2$	$=60x^2$.
Fourth differentiation,	$5\times 4\times 3\times 2x$	$=120x$.
Fifth differentiation,	$5\times 4\times 3\times 2\times 1$	$=120$.
Sixth differentiation,		$=0$.

There is a certain notation, with which we are already acquainted (see p. 15), used by some writers, that is very convenient. This is to employ the general symbol $f(x)$ for any function of x. Here the symbol $f(\ )$ is read as "function of," without saying what particular function is meant. So the statement $y=f(x)$ merely tells us that y is a function of x, it may be x^2 or ax^n, or $\cos x$ or any other complicated function of x.

The corresponding symbol for the differential coefficient is $f'(x)$, which is simpler to write than $\frac{dy}{dx}$. This is called the "derived function" of x.

Suppose we differentiate over again, we shall get the "second derived function" or second differential coefficient, which is denoted by $f''(x)$; and so on.

Now let us generalize.

Let $y=f(x)=x^n$.

First differentiation, $f'(x)=nx^{n-1}$.

Second differentiation, $f''(x)=n(n-1)x^{n-2}$.

Third differentiation, $f'''(x)=n(n-1)(n-2)x^{n-3}$.

Fourth differentiation,

$$f''''(x)=n(n-1)(n-2)(n-3)x^{n-4}.$$

etc., etc.

But this is not the only way of indicating successive differentiations. For,

if the original function be $y=f(x)$;

once differentiating gives $\frac{dy}{dx}=f'(x)$;

twice differentiating gives $\frac{d\left(\frac{dy}{dx}\right)}{dx}=f''(x)$;

and this is more conveniently written as $\frac{d^2y}{(dx)^2}$, or more usually $\frac{d^2y}{dx^2}$. Similarly, we may write as the result of thrice differentiating, $\frac{d^3y}{dx^3}=f'''(x)$.

Examples.

Now let us try $y=f(x)=7x^4+3\cdot5x^3-\frac{1}{2}x^2+x-2.$

$$\frac{dy}{dx}=f'(x)=28x^3+10\cdot5x^2-x+1,$$

$$\frac{d^2y}{dx^2}=f''(x)=84x^2+21x-1,$$

$$\frac{d^3y}{dx^3}=f'''(x)=168x+21.$$

$$\frac{d^4y}{dx^4}=f''''(x)=168,$$

$$\frac{d^5y}{dx^5}=f'''''(x)=0.$$

In a similar manner if $y=\phi(x)=3x(x^2-4)$,

$$\phi'(x)=\frac{dy}{dx}=3[x\times 2x+(x^2-4)\times 1]=3(3x^2-4),$$

$$\phi''(x)=\frac{d^2y}{dx^2}=3\times 6x=18x,$$

$$\phi'''(x)=\frac{d^3y}{dx^3}=18,$$

$$\phi''''(x)=\frac{d^4y}{dx^4}=0.$$

Exercises IV. (See page 289 for Answers.)

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following expressions:

(1) $y=17x+12x^2.$ (2) $y=\frac{x^2+a}{x+a}.$

(3) $y=1+\frac{x}{1}+\frac{x^2}{1\times 2}+\frac{x^3}{1\times 2\times 3}+\frac{x^4}{1\times 2\times 3\times 4}.$

(4) Find the 2nd and 3rd derived functions in the Exercises III. (p. 46), No. 1 to No. 7, and in the Examples given (p. 41), No. 1 to No. 7.

CHAPTER VIII.

WHEN TIME VARIES.

SOME of the most important problems of the calculus are those where time is the independent variable, and we have to think about the values of some other quantity that varies when the time varies. Some things grow larger as time goes on; some other things grow smaller. The distance that a train has travelled from its starting place goes on ever increasing as time goes on. Trees grow taller as the years go by. Which is growing at the greater rate; a plant 12 inches high which in one month becomes 14 inches high, or a tree 12 feet high which in a year becomes 14 feet high?

In this chapter we are going to make much use of the word *rate.* Nothing to do with poor-rate, or police-rate (except that even here the word suggests a proportion—a ratio—so many pence in the pound). Nothing to do even with birth-rate or death-rate, though these words suggest so many births or deaths per thousand of the population. When a motor-car whizzes by us, we say: What a terrific rate! When a spendthrift is flinging about his money, we remark that that young man is living at a prodigious rate.

What do we mean by *rate*? In both these cases we are making a mental comparison of something that is happening, and the length of time that it takes to happen. If the motor-car flies past us going 10 yards per second, a simple bit of mental arithmetic will show us that this is equivalent—while it lasts—to a rate of 600 yards per minute, or over 20 miles per hour.

Now in what sense is it true that a speed of 10 yards per second is the same as 600 yards per minute? Ten yards is not the same as 600 yards, nor is one second the same thing as one minute. What we mean by saying that the *rate* is the same, is this: that the proportion borne between distance passed over and time taken to pass over it, is the same in both cases.

Take another example. A man may have only a few pounds in his possession, and yet be able to spend money at the rate of millions a year—provided he goes on spending money at that rate for a few minutes only. Suppose you hand a shilling over the counter to pay for some goods; and suppose the operation lasts exactly one second. Then, during that brief operation, you are parting with your money at the rate of 1 shilling per second, which is the same rate as £3 per minute, or £180 per hour, or £4320 per day, or £1,576,800 per year! If you have £10 in your pocket, you can go on spending money at the rate of a million a year for just $5\frac{1}{4}$ minutes.

It is said that Sandy had not been in London above five minutes when "bang went saxpence." If he were to spend money at that rate throughout a day of 12 hours, he would be spending 6 shillings an hour, or £3. 12*s.* per day, or £21. 12*s.* a week, not counting the Sawbath.

Now try to put some of these ideas into differential notation.

Let y in this case stand for money, and let t stand for time.

If you are spending money, and the amount you spend in a short time dt be called dy, the *rate* of spending it will be $\dfrac{dy}{dt}$; or, as regards saving, with a minus sign, as $-\dfrac{dy}{dt}$, because then dy is a *decrement*, not an increment. But money is not a good example for the calculus, because it generally comes and goes by jumps, not by a continuous flow—you may earn £200 a year, but it does not keep running in all day long in a thin stream; it comes in only weekly, or monthly, or quarterly, in lumps: and your expenditure also goes out in sudden payments.

A more apt illustration of the idea of a rate is furnished by the speed of a moving body. From London (Euston station) to Liverpool is 200 miles. If a train leaves London at 7 o'clock, and reaches Liverpool at 11 o'clock, you know that, since it has travelled 200 miles in 4 hours, its average rate must have been 50 miles per hour; because $\frac{200}{4} = \frac{50}{1}$. Here you are really making a mental comparison between the distance passed over and the time taken to pass over it. You are dividing one by the other. If y is the whole distance, and t the whole time, clearly the average rate is $\frac{y}{t}$. Now the speed was not actually constant all the way: at starting, and during the slowing up at the end of the journey, the speed was less. Probably at some part, when running downhill, the speed was over 60 miles an hour. If, during any particular element of time dt, the corresponding element of distance passed over was dy, then at that part of the journey the speed was $\frac{dy}{dt}$. The *rate* at which one quantity (in the present instance, *distance*) is changing in relation to the other quantity (in this case, *time*) is properly expressed, then, by stating the differential coefficient of one with respect to the other. A *velocity*, scientifically expressed, is the rate at which a very small distance in any given direction is being passed over; and may therefore be written

$$v=\frac{dy}{dt}.$$

But if the velocity v is not uniform, then it must be either increasing or else decreasing. The rate at which a velocity is increasing is called the *acceleration*. If a moving body is, at any particular instant, gaining an additional velocity dv in an element of time dt, then the acceleration a at that instant may be written

$$a=\frac{dv}{dt};$$

but dv is itself $d\left(\frac{dy}{dt}\right)$. Hence we may put

$$a=\frac{d\left(\frac{dy}{dt}\right)}{dt};$$

and this is usually written $a=\frac{d^2y}{dt^2}$;

or the acceleration is the second differential coefficient of the distance, with respect to time. Acceleration is expressed as a change of velocity in unit time, for instance, as being so many feet per second per second; the notation used being feet ÷ second^2.

When a railway train has just begun to move, its velocity v is small; but it is rapidly gaining speed—it is being hurried up, or accelerated, by the effort of the engine. So its $\frac{d^2y}{dt^2}$ is large. When it has got up its top speed it is no longer being accelerated, so that then $\frac{d^2y}{dt^2}$ has fallen to zero. But when it nears its stopping place its speed begins to slow down; may, indeed, slow down very quickly if the brakes are put on, and during this period of *deceleration* or slackening of pace, the value of $\frac{dv}{dt}$, that is, of $\frac{d^2y}{dt^2}$, will be negative.

To accelerate a mass m requires the continuous application of force. The force necessary to accelerate a mass is proportional to the mass, and it is also proportional to the acceleration which is being imparted. Hence we may write for the force f, the expression

$$f=ma;$$

or $$f = m\frac{dv}{dt};$$

or $$f = m\frac{d^2y}{dt^2}.$$

The product of a mass by the speed at which it is going is called its *momentum*, and is in symbols mv. If we differentiate momentum with respect to time we shall get $\frac{d(mv)}{dt}$ for the rate of change of momentum. But, since m is a constant quantity, this may be written $m\frac{dv}{dt}$, which we see above is the same as f. That is to say, force may be expressed either as mass times acceleration, or as rate of change of momentum.

Again, if a force is employed to move something (against an equal and opposite counter-force), it does *work*; and the amount of work done is measured by the product of the force into the distance (in its own direction) through which its point of application moves forward. So if a force f moves forward through a length y, the work done (which we may call w) will be

$$w = f \times y;$$

where we take f as a constant force. If the force varies at different parts of the range y, then we must find an expression for its value from point to point. If f be the force along the small element of length dy, the amount of work done will be $f \times dy$. But as dy is only an element of length, only an element of work will be done. If we write w for work, then an element of work will be dw; and we have

$$dw = f \times dy;$$

which may be written

$$dw = ma \cdot dy;$$

or

$$dw = m\frac{d^2y}{dt^2} \cdot dy$$

or

$$dw = m\frac{dv}{dt} \cdot dy.$$

Further, we may transpose the expression and write

$$\frac{dw}{dy} = f.$$

This gives us yet a third definition of *force*; that if it is being used to produce a displacement in any direction, the force (in that direction) is equal to the rate at which work is being done per unit of length in that direction. In this last sentence the word *rate* is clearly not used in its time-sense, but in its meaning as ratio or proportion.

Sir Isaac Newton, who was (along with Leibnitz) an inventor of the methods of the calculus, regarded all quantities that were varying as *flowing*; and the ratio which we nowadays call the differential coefficient he regarded as the rate of flowing, or the *fluxion* of the quantity in question. He did not use the notation of the dy and dx, and dt (this was due to Leibnitz), but had instead a notation of his own. If y was a quantity that varied, or "flowed," then his symbol for its rate of variation (or "fluxion") was

$\dot{y}$. If x was the variable, then its fluxion was called $\dot{x}$. The dot over the letter indicated that it had been differentiated. But this notation does not tell us what is the independent variable with respect to which the differentiation has been effected. When we see $\frac{dy}{dt}$ we know that y is to be differentiated with respect to t. If we see $\frac{dy}{dx}$ we know that y is to be differentiated with respect to x. But if we see merely $\dot{y}$, we cannot tell without looking at the context whether this is to mean $\frac{dy}{dx}$ or $\frac{dy}{dt}$ or $\frac{dy}{dz}$, or what is the other variable. So, therefore, this fluxional notation is less informing than the differential notation, and has in consequence largely dropped out of use. But its simplicity gives it an advantage if only we will agree to use it for those cases exclusively where *time* is the independent variable. In that case $\dot{y}$ will mean $\frac{dy}{dt}$ and $\dot{u}$ will mean $\frac{du}{dt}$; and $\ddot{x}$ will mean $\frac{d^2x}{dt^2}$.

Adopting this fluxional notation we may write the mechanical equations considered in the paragraphs above, as follows:

distance	x
velocity	$v=\dot{x}$,
acceleration	$a=\dot{v}=\ddot{x}$,
force	$f=m\dot{v}=m\ddot{x}$,
work	$w=x\times m\ddot{x}$.

Examples.

(1) A body moves so that the distance x (in feet), which it travels from a certain point O, is given by the relation $x=0{\cdot}2t^2+10{\cdot}4$, where t is the time in seconds elapsed since a certain instant. Find the velocity and acceleration 5 seconds after the body began to move, and also find the corresponding values when the distance covered is 100 feet. Find also the average velocity during the first 10 seconds of its motion. (Suppose distances and motion to the right to be positive.)

Now $$x=0{\cdot}2t^2+10{\cdot}4,$$

$$v=\dot{x}=\frac{dx}{dt}=0{\cdot}4t; \text{ and } a=\ddot{x}=\frac{d^2x}{dt^2}=0{\cdot}4=\text{constant.}$$

When $t=0$, $x=10{\cdot}4$ and $v=0$. The body started from a point 10·4 feet to the right of the point O: and the time was reckoned from the instant the body started.

When $t=5$, $v=0{\cdot}4\times 5=2$ ft./sec.; $a=0{\cdot}4$ ft./sec².

When $x=100$, $100=0{\cdot}2t^2+10{\cdot}4$, or $t^2=448$,

and $t=21{\cdot}17$ sec.; $v=0{\cdot}4\times 21{\cdot}17=8{\cdot}468$ ft./sec.

When $t=10$,

distance travelled $=0{\cdot}2\times 10^2+10{\cdot}4-10{\cdot}4=20$ ft.

Average velocity $=\frac{20}{10}=2$ ft./sec.

(It is the same velocity as the velocity at the middle of the interval, $t=5$; for, the acceleration being constant, the velocity has varied uniformly from zero when $t=0$ to 4 ft./sec. when $t=10$.)

(2) In the above problem let us suppose

$$x = 0{\cdot}2t^2 + 3t + 10{\cdot}4.$$

$$v = \dot{x} = \frac{dx}{dt} = 0{\cdot}4t + 3\,; \quad a = \ddot{x} = \frac{d^2x}{dt^2} = 0{\cdot}4 = \text{constant.}$$

When $t = 0$, $x = 10{\cdot}4$ and $v = 3$ ft./sec., the time is reckoned from the instant at which the body passed a point 10·4 ft. from the point O, its velocity being then already 3 ft./sec. To find the time elapsed since it began moving, let $v = 0$; then $0{\cdot}4t + 3 = 0$, $t = -\frac{3}{\cdot 4} = -7{\cdot}5$ sec. The body began moving 7·5 sec. before time was begun to be observed; 5 seconds after this gives $t = -2{\cdot}5$ and $v = 0{\cdot}4 \times -2{\cdot}5 + 3 = 2$ ft./sec.

When $x = 100$ ft.,

$$100 = 0{\cdot}2t^2 + 3t + 10{\cdot}4\,; \text{ or } t^2 + 15t - 448 = 0;$$

hence $t = 14{\cdot}95$ sec., $v = 0{\cdot}4 \times 14{\cdot}95 + 3 = 8{\cdot}98$ ft./sec.

To find the distance travelled during the 10 first seconds of the motion one must know how far the body was from the point O when it started.

When $t = -7{\cdot}5$,

$$x = 0{\cdot}2 \times (-7{\cdot}5)^2 - 3 \times 7{\cdot}5 + 10{\cdot}4 = -0{\cdot}85 \text{ ft.},$$

that is 0·85 ft. to the left of the point O.

Now, when $t = 2{\cdot}5$,

$$x = 0{\cdot}2 \times 2{\cdot}5^2 + 3 \times 2{\cdot}5 + 10{\cdot}4 = 19{\cdot}15.$$

So, in 10 seconds, the distance travelled was $19{\cdot}15 + 0{\cdot}85 = 20$ ft., and

$$\text{the average velocity} = \tfrac{20}{10} = 2 \text{ ft./sec.}$$

(3) Consider a similar problem when the distance is given by $x = 0{\cdot}2t^2 - 3t + 10{\cdot}4$. Then $v = 0{\cdot}4t - 3$, $a = 0{\cdot}4 = \text{constant}$. When $t = 0$, $x = 10{\cdot}4$ as before, and

$v=-3$; so that the body was moving in the direction opposite to its motion in the previous cases. As the acceleration is positive, however, we see that this velocity will decrease as time goes on, until it becomes zero, when $v=0$ or $0{\cdot}4t-3=0$; or $t=7{\cdot}5$ sec. After this, the velocity becomes positive; and 5 seconds after the body started, $t=12{\cdot}5$, and

$$v=0{\cdot}4\times 12{\cdot}5-3=2 \text{ ft./sec.}$$

When $x=100$,

$$100=0{\cdot}2t^2-3t+10{\cdot}4, \quad \text{or} \quad t^2-15t-448=0,$$

and $\quad t=29{\cdot}95$; $v=0{\cdot}4\times 29{\cdot}95-3=8{\cdot}98$ ft./sec.

When v is zero, $x=0{\cdot}2\times 7{\cdot}5^2-3\times 7{\cdot}5+10{\cdot}4=-0{\cdot}85$, informing us that the body moves back to $0{\cdot}85$ ft. beyond the point O before it stops. Ten seconds later $t=17{\cdot}5$ and $x=0{\cdot}2\times 17{\cdot}5^2-3\times 17{\cdot}5+10{\cdot}4=19{\cdot}15$. The distance travelled $={\cdot}85+19{\cdot}15=20{\cdot}0$, and the average velocity is again 2 ft./sec.

(4) Consider yet another problem of the same sort with $x=0{\cdot}2t^3-3t^2+10{\cdot}4$; $v=0{\cdot}6t^2-6t$; $a=1{\cdot}2t-6$. The acceleration is no more constant.

When $t=0$, $x=10{\cdot}4$, $v=0$, $a=-6$. The body is at rest, but just ready to move with a negative acceleration, that is to gain a velocity towards the point O.

(5) If we have $x=0{\cdot}2t^3-3t+10{\cdot}4$, then $v=0{\cdot}6t^2-3$, and $a=1{\cdot}2t$.

When $t=0$, $x=10{\cdot}4$; $v=-3$; $a=0$.

The body is moving towards the point O with a velocity of 3 ft./sec., and just at that instant the velocity is uniform.

We see that the conditions of the motion can always be at once ascertained from the time-distance equation and its first and second derived functions. In the last two cases the mean velocity during the first 10 seconds and the velocity 5 seconds after the start will no more be the same, because the velocity is not increasing uniformly, the acceleration being no longer constant.

(6) The angle θ (in radians) turned through by a wheel is given by $\theta=3+2t-0{\cdot}1t^3$, where t is the time in seconds from a certain instant; find the angular velocity ω and the angular acceleration α, (a) after 1 second; (b) after it has performed one revolution. At what time is it at rest, and how many revolutions has it performed up to that instant?

Writing for the acceleration

$$\omega=\dot{\theta}=\frac{d\theta}{dt}=2-0{\cdot}3t^2,\ \alpha=\ddot{\theta}=\frac{d^2\theta}{dt^2}=-0{\cdot}6t.$$

When $t=0$, $\theta=3$; $\omega=2$ rad./sec.; $\alpha=0$.

When $t=1$,

$$\omega=2-0{\cdot}3=1{\cdot}7 \text{ rad./sec.};\ \alpha=-0{\cdot}6 \text{ rad./sec}^2.$$

This is a retardation; the wheel is slowing down.

After 1 revolution

$$\theta=2\pi=6{\cdot}28;\ 6{\cdot}28=3+2t-0{\cdot}1t^3.$$

By plotting the graph, $\theta=3+2t-0{\cdot}1t^3$, we can get the value or values of t for which $\theta=6{\cdot}28$; these are 2·11 and 3·03 (there is a third negative value).

When $t=2{\cdot}11$,

$$\theta=6{\cdot}28;\ \omega=2-1{\cdot}34=0{\cdot}66 \text{ rad./sec.};$$
$$\alpha=-1{\cdot}27 \text{ rad./sec}^2.$$

When $t=3{\cdot}03$,

$$\theta=6{\cdot}28;\ \omega=2-2{\cdot}754=-0{\cdot}754 \text{ rad./sec.};$$
$$\alpha=-1{\cdot}82 \text{ rad./sec}^2.$$

The velocity is reversed. The wheel is evidently at rest between these two instants; it is at rest when $\omega=0$, that is when $0=2-0{\cdot}3t^2$, or when $t=2{\cdot}58$ sec., it has performed

$$\frac{\theta}{2\pi}=\frac{3+2\times 2{\cdot}58-0\,1\times 2{\cdot}58^3}{6{\cdot}28}=1{\cdot}025 \text{ revolutions.}$$

Exercises V. (See page 290 for Answers.)

(1) If $y=a+bt^2+ct^4$; find $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$.

$$Ans.\quad \frac{dy}{dt}=2bt+4ct^3;\ \frac{d^2y}{dt^2}=2b+12ct^2.$$

(2) A body falling freely in space describes in t seconds a space s, in feet, expressed by the equation $s=16t^2$. Draw a curve showing the relation between s and t. Also determine the velocity of the body at the following times from its being let drop: $t=2$ seconds; $t=4{\cdot}6$ seconds; $t=0{\cdot}01$ second.

(3) If $x=at-\frac{1}{2}gt^2$; find $\dot{x}$ and $\ddot{x}$.

(4) If a body move according to the law

$$s=12-4{\cdot}5t+6{\cdot}2t^2,$$

find its velocity when $t=4$ seconds; s being in feet.

(5) Find the acceleration of the body mentioned in the preceding example. Is the acceleration the same for all values of t?

(6) The angle θ (in radians) turned through by a revolving wheel is connected with the time t (in seconds) that has elapsed since starting, by the law

$$\theta = 2{\cdot}1 - 3{\cdot}2t + 4{\cdot}8t^2.$$

Find the angular velocity (in radians per second) of that wheel when $1\frac{1}{2}$ seconds have elapsed. Find also its angular acceleration.

(7) A slider moves so that, during the first part of its motion, its distance s in inches from its starting point is given by the expression

$$s = 6{\cdot}8t^3 - 10{\cdot}8t; \quad t \text{ being in seconds.}$$

Find the expression for the velocity and the acceleration at any time; and hence find the velocity and the acceleration after 3 seconds.

(8) The motion of a rising balloon is such that its height h, in miles, is given at any instant by the expression $h = 0{\cdot}5 + \frac{1}{10}\sqrt[3]{t - 125}$; t being in seconds.

Find an expression for the velocity and the acceleration at any time. Draw curves to show the variation of height, velocity and acceleration during the first ten minutes of the ascent.

(9) A stone is thrown downwards into water and its depth p in metres at any instant t seconds after reaching the surface of the water is given by the expression

$$p = \frac{4}{4 + t^2} + 0{\cdot}8t - 1.$$

Find an expression for the velocity and the acceleration at any time. Find the velocity and acceleration after 10 seconds.

(10) A body moves in such a way that the spaces described in the time t from starting is given by $s = t^n$, where n is a constant. Find the value of n when the velocity is doubled from the 5th to the 10th second; find it also when the velocity is numerically equal to the acceleration at the end of the 10th second.

CHAPTER IX.

INTRODUCING A USEFUL DODGE.

SOMETIMES one is stumped by finding that the expression to be differentiated is too complicated to tackle directly.

Thus, the equation

$$y=(x^2+a^2)^{\frac{3}{2}}$$

is awkward to a beginner.

Now the dodge to turn the difficulty is this: Write some symbol, such as u, for the expression x^2+a^2; then the equation becomes

$$y=u^{\frac{3}{2}},$$

which you can easily manage; for

$$\frac{dy}{du}=\frac{3}{2}u^{\frac{1}{2}}.$$

Then tackle the expression

$$u=x^2+a^2,$$

and differentiate it with respect to x

$$\frac{du}{dx}=2x.$$

Then all that remains is plain sailing;

for $$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx};$$

that is, $$\frac{dy}{dx}=\frac{3}{2}u^{\frac{1}{2}}\times 2x$$
$$=\tfrac{3}{2}(x^2+a^2)^{\frac{1}{2}}\times 2x$$
$$=3x(x^2+a^2)^{\frac{1}{2}};$$

and so the trick is done.

By and bye, when you have learned how to deal with sines, and cosines, and exponentials, you will find this dodge of increasing usefulness.

Examples.

Let us practise this dodge on a few examples.

(1) Differentiate $y=\sqrt{a+x}$.

Let $a+x=u$.

$$\frac{du}{dx}=1;\quad y=u^{\frac{1}{2}};\quad \frac{dy}{du}=\tfrac{1}{2}u^{-\frac{1}{2}}=\tfrac{1}{2}(a+x)^{-\frac{1}{2}}.$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2\sqrt{a+x}}.$$

(2) Differentiate $y=\dfrac{1}{\sqrt{a+x^2}}$.

Let $a+x^2=u$.

$$\frac{du}{dx}=2x;\quad y=u^{-\frac{1}{2}};\quad \frac{dy}{du}=-\tfrac{1}{2}u^{-\frac{3}{2}}.$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-\frac{x}{\sqrt{(a+x^2)^3}}.$$

(3) Differentiate $y=\left(m-nx^{\frac{2}{3}}+\dfrac{p}{x^{\frac{4}{3}}}\right)^a.$

Let $m-nx^{\frac{2}{3}}+px^{-\frac{4}{3}}=u.$

$$\frac{du}{dx}=-\tfrac{2}{3}nx^{-\frac{1}{3}}-\tfrac{4}{3}px^{-\frac{7}{3}};$$

$$y=u^a;\ \frac{dy}{du}=au^{a-1}.$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-a\left(m-nx^{\frac{2}{3}}+\frac{p}{x^{\frac{4}{3}}}\right)^{a-1}\left(\tfrac{2}{3}nx^{-\frac{1}{3}}+\tfrac{4}{3}px^{-\frac{7}{3}}\right).$$

(4) Differentiate $y=\dfrac{1}{\sqrt{x^3-a^2}}.$

Let $u=x^3-a^2.$

$$\frac{du}{dx}=3x^2;\ y=u^{-\frac{1}{2}};\ \frac{dy}{du}=-\tfrac{1}{2}(x^3-a^2)^{-\frac{3}{2}}.$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-\frac{3x^2}{2\sqrt{(x^3-a^2)^3}}.$$

(5) Differentiate $y=\sqrt{\dfrac{1-x}{1+x}}.$

Write this as $y=\dfrac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}.$

$$\frac{dy}{dx}=\frac{(1+x)^{\frac{1}{2}}\dfrac{d(1-x)^{\frac{1}{2}}}{dx}-(1-x)^{\frac{1}{2}}\dfrac{d(1+x)^{\frac{1}{2}}}{dx}}{1+x}.$$

(We may also write $y=(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$ and differentiate as a product.)

Proceeding as in exercise (1) above, we get

$$\frac{d(1-x)^{\frac{1}{2}}}{dx} = -\frac{1}{2\sqrt{1-x}}; \quad \text{and} \quad \frac{d(1+x)^{\frac{1}{2}}}{dx} = \frac{1}{2\sqrt{1+x}}.$$

Hence

$$\frac{dy}{dx} = -\frac{(1+x)^{\frac{1}{2}}}{2(1+x)\sqrt{1-x}} - \frac{(1-x)^{\frac{1}{2}}}{2(1+x)\sqrt{1+x}}$$

$$= -\frac{1}{2\sqrt{1+x}\sqrt{1-x}} - \frac{\sqrt{1-x}}{2\sqrt{(1+x)^3}};$$

or $$\frac{dy}{dx} = -\frac{1}{(1+x)\sqrt{1-x^2}}.$$

(6) Differentiate $y = \sqrt{\dfrac{x^3}{1+x^2}}$.

We may write this

$$y = x^{\frac{3}{2}}(1+x^2)^{-\frac{1}{2}};$$

$$\frac{dy}{dx} = \tfrac{3}{2}x^{\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} + x^{\frac{3}{2}} \times \frac{d[(1+x^2)^{-\frac{1}{2}}]}{dx}.$$

Differentiating $(1+x^2)^{-\frac{1}{2}}$, as shown in exercise (2) above, we get

$$\frac{d[(1+x^2)^{-\frac{1}{2}}]}{dx} = -\frac{x}{\sqrt{(1+x^2)^3}};$$

so that

$$\frac{dy}{dx} = \frac{3\sqrt{x}}{2\sqrt{1+x^2}} - \frac{\sqrt{x^5}}{\sqrt{(1+x^2)^3}} = \frac{\sqrt{x}(3+x^2)}{2\sqrt{(1+x^2)^3}}.$$

(7) Differentiate $y=(x+\sqrt{x^2+x+a})^3$.

Let $x+\sqrt{x^2+x+a}=u$.

$$\frac{du}{dx}=1+\frac{d[(x^2+x+a)^{\frac{1}{2}}]}{dx}.$$

$$y=u^3;\text{ and }\frac{dy}{du}=3u^2=3(x+\sqrt{x^2+x+a})^2.$$

Now let $(x^2+x+a)^{\frac{1}{2}}=v$ and $(x^2+x+a)=w$.

$$\frac{dw}{dx}=2x+1;\quad v=w^{\frac{1}{2}};\quad \frac{dv}{dw}=\tfrac{1}{2}w^{-\frac{1}{2}}.$$

$$\frac{dv}{dx}=\frac{dv}{dw}\times\frac{dw}{dx}=\tfrac{1}{2}(x^2+x+a)^{-\frac{1}{2}}(2x+1).$$

Hence $\dfrac{du}{dx}=1+\dfrac{2x+1}{2\sqrt{x^2+x+a}}$,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=3(x+\sqrt{x^2+x+a})^2\left(1+\frac{2x+1}{2\sqrt{x^2+x+a}}\right).$$

(8) Differentiate $y=\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}\sqrt[3]{\dfrac{a^2-x^2}{a^2+x^2}}$.

We get

$$y=\frac{(a^2+x^2)^{\frac{1}{2}}(a^2-x^2)^{\frac{1}{3}}}{(a^2-x^2)^{\frac{1}{2}}(a^2+x^2)^{\frac{1}{3}}}=(a^2+x^2)^{\frac{1}{6}}(a^2-x^2)^{-\frac{1}{6}}.$$

$$\frac{dy}{dx}=(a^2+x^2)^{\frac{1}{6}}\frac{d[(a^2-x^2)^{-\frac{1}{6}}]}{dx}+\frac{d[(a^2+x^2)^{\frac{1}{6}}]}{(a^2-x^2)^{\frac{1}{6}}dx}.$$

Let $u=(a^2-x^2)^{-\frac{1}{6}}$ and $v=(a^2-x^2)$.

$$u=v^{-\frac{1}{6}};\quad \frac{du}{dv}=-\frac{1}{6}v^{-\frac{7}{6}};\quad \frac{dv}{dx}=-2x.$$

$$\frac{du}{dx}=\frac{du}{dv}\times\frac{dv}{dx}=\frac{1}{3}x(a^2-x^2)^{-\frac{7}{6}}.$$

Let $w=(a^2+x^2)^{\frac{1}{6}}$ and $z=(a^2+x^2)$.

$$w=z^{\frac{1}{6}};\quad \frac{dw}{dz}=\frac{1}{6}z^{-\frac{5}{6}};\quad \frac{dz}{dx}=2x.$$

$$\frac{dw}{dx}=\frac{dw}{dz}\times\frac{dz}{dx}=\frac{1}{3}x(a^2+x^2)^{-\frac{5}{6}}.$$

Hence

$$\frac{dy}{dx}=(a^2+x^2)^{\frac{1}{6}}\frac{x}{3(a^2-x^2)^{\frac{7}{6}}}+\frac{x}{3(a^2-x^2)^{\frac{1}{6}}(a^2+x^2)^{\frac{5}{6}}};$$

or $$\frac{dy}{dx}=\frac{x}{3}\left[\sqrt[6]{\frac{a^2+x^2}{(a^2-x^2)^7}}+\frac{1}{\sqrt[6]{(a^2-x^2)(a^2+x^2)^5}}\right].$$

(9) Differentiate y^n with respect to y^5.

$$\frac{d(y^n)}{d(y^5)}=\frac{ny^{n-1}}{5y^{5-1}}=\frac{n}{5}y^{n-5}.$$

(10) Find the first and second differential coefficients of $y=\frac{x}{b}\sqrt{(a-x)x}$.

$$\frac{dy}{dx}=\frac{x}{b}\,\frac{d\{[(a-x)x]^{\frac{1}{2}}\}}{dx}+\frac{\sqrt{(a-x)x}}{b}.$$

Let $[(a-x)x]^{\frac{1}{2}}=u$ and let $(a-x)x=w$; then $u=w^{\frac{1}{2}}$.

$$\frac{du}{dw}=\frac{1}{2}w^{-\frac{1}{2}}=\frac{1}{2w^{\frac{1}{2}}}=\frac{1}{2\sqrt{(a-x)x}}.$$

$$\frac{dw}{dx}=a-2x.$$

$$\frac{du}{dw}\times\frac{dw}{dx}=\frac{du}{dx}=\frac{a-2x}{2\sqrt{(a-x)x}}.$$

Hence

$$\frac{dy}{dx}=\frac{x(a-2x)}{2b\sqrt{(a-x)x}}+\frac{\sqrt{(a-x)x}}{b}=\frac{x(3a-4x)}{2b\sqrt{(a-x)x}}.$$

Now

$$\frac{d^2y}{dx^2}=\frac{2b\sqrt{(a-x)x}(3a-8x)-\dfrac{(3ax-4x^2)b(a-2x)}{\sqrt{(a-x)x}}}{4b^2(a-x)x}$$

$$=\frac{3a^2-12ax+8x^2}{4b(a-x)\sqrt{(a-x)x}}.$$

(We shall need these two last differential coefficients later on. See Ex. X. No. 11.)

Exercises VI. (See page 291 for Answers.)

Differentiate the following:

(1) $y=\sqrt{x^2+1}.$ (2) $y=\sqrt{x^2+a^2}.$

(3) $y=\dfrac{1}{\sqrt{a+x}}.$ (4) $y=\dfrac{a}{\sqrt{a-x^2}}.$

(5) $y=\dfrac{\sqrt{x^2-a^2}}{x^2}.$ (6) $y=\dfrac{\sqrt[3]{x^4+a}}{\sqrt[2]{x^3+a}}.$

(7) $y=\dfrac{a^2+x^2}{(a+x)^2}.$

(8) Differentiate y^5 with respect to y^2.

(9) Differentiate $y=\dfrac{\sqrt{1-\theta^2}}{1-\theta}$.

The process can be extended to three or more differential coefficients, so that $\dfrac{dy}{dx}=\dfrac{dy}{dz}\times\dfrac{dz}{dv}\times\dfrac{dv}{dx}$.

Examples.

(1) If $z=3x^4$; $v=\dfrac{7}{z^2}$; $y=\sqrt{1+v}$, find $\dfrac{dy}{dx}$.

We have

$$\frac{dy}{dv}=\frac{1}{2\sqrt{1+v}};\quad \frac{dv}{dz}=-\frac{14}{z^3};\quad \frac{dz}{dx}=12x^3.$$

$$\frac{dy}{dx}=-\frac{168x^3}{(2\sqrt{1+v})z^3}=-\frac{28}{3x^5\sqrt{9x^8+7}}.$$

(2) If $t=\dfrac{1}{5\sqrt{\theta}}$; $x=t^3+\dfrac{t}{2}$; $v=\dfrac{7x^2}{\sqrt[3]{x-1}}$, find $\dfrac{dv}{d\theta}$.

$$\frac{dv}{dx}=\frac{7x(5x-6)}{3\sqrt[3]{(x-1)^4}};\quad \frac{dx}{dt}=3t^2+\tfrac{1}{2};\quad \frac{dt}{d\theta}=-\frac{1}{10\sqrt{\theta^3}}.$$

Hence $$\frac{dv}{d\theta}=-\frac{7x(5x-6)(3t^2+\frac{1}{2})}{30\sqrt[3]{(x-1)^4}\sqrt{\theta^3}},$$

an expression in which x must be replaced by its value, and t by its value in terms of θ.

(3) If $\theta=\dfrac{3a^2x}{\sqrt{x^3}}$; $\omega=\dfrac{\sqrt{1-\theta^2}}{1+\theta}$; and $\phi=\sqrt{3}-\dfrac{1}{\omega\sqrt{2}}$, find $\dfrac{d\phi}{dx}$.

We get

$$\theta=3a^2x^{-\frac{1}{2}};\quad \omega=\sqrt{\frac{1-\theta}{1+\theta}};\quad \text{and}\quad \phi=\sqrt{3}-\frac{1}{\sqrt{2}}\omega^{-1}.$$

$$\frac{d\theta}{dx}=-\frac{3a^2}{2\sqrt{x^3}};\quad \frac{d\omega}{d\theta}=-\frac{1}{(1+\theta)\sqrt{1-\theta^2}}$$

(see example 5, p. 69); and

$$\frac{d\phi}{d\omega}=\frac{1}{\sqrt{2}\omega^2}.$$

So that $\dfrac{d\phi}{dx}=\dfrac{1}{\sqrt{2}\times\omega^2}\times\dfrac{1}{(1+\theta)\sqrt{1-\theta^2}}\times\dfrac{3a^2}{2\sqrt{x^3}}$

Replace now first ω, then θ by its value.

Exercises VII.

You can now successfully try the following. (See page 291 for Answers.)

(1) If $u=\frac{1}{2}x^3$; $v=3(u+u^2)$; and $w=\dfrac{1}{v^2}$, find $\dfrac{dw}{dx}$.

(2) If $y=3x^2+\sqrt{2}$; $z=\sqrt{1+y}$; and $v=\dfrac{1}{\sqrt{3}+4z}$, find $\dfrac{dv}{dx}$.

(3) If $y=\dfrac{x^3}{\sqrt{3}}$; $z=(1+y)^2$; and $u=\dfrac{1}{\sqrt{1+z}}$, find $\dfrac{du}{dx}$.

CHAPTER X.

GEOMETRICAL MEANING OF DIFFERENTIATION.

IT is useful to consider what geometrical meaning can be given to the differential coefficient.

In the first place, any function of x, such, for example, as x^2, or $\sqrt{x}$, or $ax+b$, can be plotted as a curve; and nowadays every schoolboy is familiar with the process of curve-plotting.

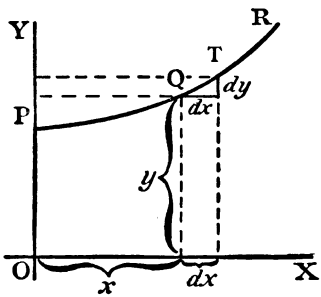

FIG. 7.

Let PQR, in Fig. 7, be a portion of a curve plotted with respect to the axes of coordinates OX and OY. Consider any point Q on this curve, where the abscissa of the point is x and its ordinate is y. Now observe how y changes when x is varied. If x

is made to increase by a small increment dx, to the right, it will be observed that y also (in *this* particular curve) increases by a small increment dy (because this particular curve happens to be an *ascending* curve). Then the ratio of dy to dx is a measure of the degree to which the curve is sloping up between the two points Q and T. As a matter of fact, it can be seen on the figure that the curve between Q and T has many different slopes, so that we cannot very well speak of the slope of the curve between Q and T. If, however, Q and T are so near each other that the small portion QT of the curve is practically straight, then it is true to say that the ratio $\frac{dy}{dx}$ is the slope of the curve along QT. The straight line QT produced on either side touches the curve along the portion QT only, and if this portion is indefinitely small, the straight line will touch the curve at practically one point only, and be therefore a *tangent* to the curve.

This tangent to the curve has evidently the same slope as QT, so that $\frac{dy}{dx}$ is the slope of the tangent to the curve at the point Q for which the value of $\frac{dy}{dx}$ is found.

We have seen that the short expression "the slope of a curve" has no precise meaning, because a curve has so many slopes—in fact, every small portion of a curve has a different slope. "The slope of a curve *at a point*" is. however, a perfectly defined thing; it is the slope of a very small portion of the curve situated just at that point; and we have seen that this is the same as "the slope of the tangent to the curve at that point."

Observe that dx is a short step to the right, and dy the corresponding short step upwards. These steps must be considered as short as possible—in fact indefinitely short,—though in diagrams we have to represent them by bits that are not infinitesimally small, otherwise they could not be seen.

We shall hereafter make considerable use of this circumstance that $\frac{dy}{dx}$ *represents the slope of the curve at any point.*

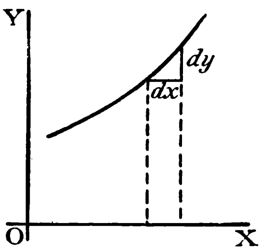

FIG. 8.

If a curve is sloping up at 45° at a particular point, as in Fig. 8, dy and dx will be equal, and the value of $\frac{dy}{dx} = 1$.

If the curve slopes up steeper than 45° (Fig. 9), $\frac{dy}{dx}$ will be greater than 1.

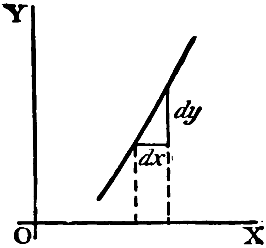

FIG. 9.

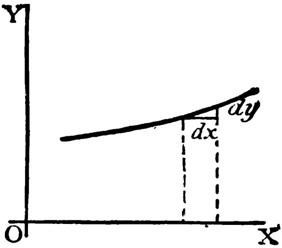

FIG. 10.

If the curve slopes up very gently, as in Fig. 10, $\frac{dy}{dx}$ will be a fraction smaller than 1.

For a horizontal line, or a horizontal place in a curve, $dy = 0$, and therefore $\frac{dy}{dx} = 0$.

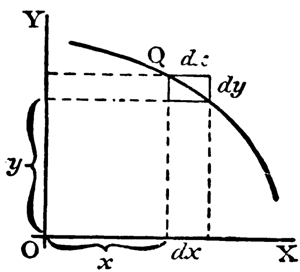

FIG. 11.

If a curve slopes *downward*, as in Fig. 11, dy will be a step down, and must therefore be reckoned of negative value; hence $\frac{dy}{dx}$ will have negative sign also.

If the "curve" happens to be a straight line, like that in Fig. 12, the value of $\frac{dy}{dx}$ will be the same at all points along it. In other words its *slope* is constant.

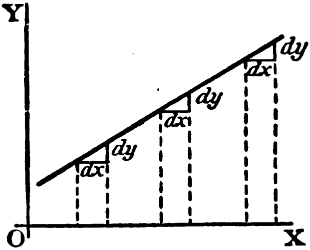

FIG. 12.

If a curve is one that turns more upwards as it goes along to the right, the values of $\frac{dy}{dx}$ will become

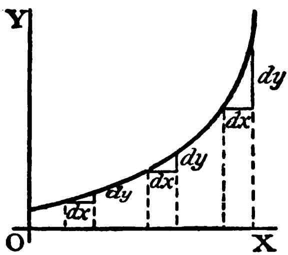

FIG. 13.

greater and greater with the increasing steepness, as in Fig. 13.

If a curve is one that gets flatter and flatter as it goes along, the values of $\frac{dy}{dx}$ will become smaller and smaller as the flatter part is reached, as in Fig. 14.

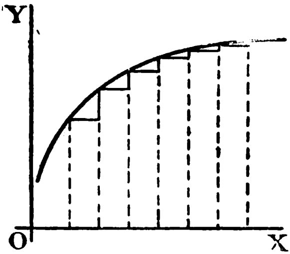

FIG. 14.

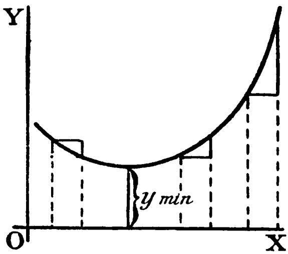

FIG. 15.

If a curve first descends, and then goes up again, as in Fig. 15, presenting a concavity upwards, then clearly $\frac{dy}{dx}$ will first be negative, with diminishing values as the curve flattens, then will be zero at the point where the bottom of the trough of the curve is reached; and from this point onward $\frac{dy}{dx}$ will have positive values that go on increasing. In such a case y is said to pass through a *minimum*. The minimum value of y is not necessarily the smallest value of y, it is that value of y corresponding to the bottom of the trough; for instance, in Fig. 28 (p. 101), the value of y corresponding to the bottom of the trough is 1, while y takes elsewhere values which are smaller than this. The characteristic of a minimum is that y must increase *on either side* of it.

N.B.—For the particular value of x that makes y a *minimum*, the value of $\frac{dy}{dx}=0$.

If a curve first ascends and then descends, the values of $\frac{dy}{dx}$ will be positive at first; then zero, as the summit is reached; then negative, as the curve slopes downwards, as in Fig. 16. In this case y is said to pass through a *maximum*, but the maximum value of y is not necessarily the greatest value of y. In Fig. 28, the maximum of y is $2\frac{1}{3}$, but this is by no means the greatest value y can have at some other point of the curve.

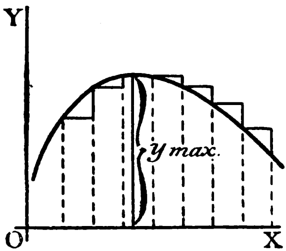

FIG. 16.

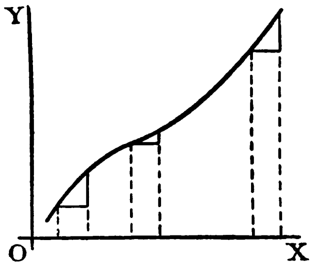

FIG. 17.

N.B.—For the particular value of x that makes y a *maximum*, the value of $\frac{dy}{dx}=0$.

If a curve has the particular form of Fig. 17, the values of $\frac{dy}{dx}$ will always be positive; but there will be one particular place where the slope is least steep, where the value of $\frac{dy}{dx}$ will be a minimum; that is, less than it is at any other part of the curve.

If a curve has the form of Fig. 18, the value of $\frac{dy}{dx}$ will be negative in the upper part, and positive in the lower part; while at the nose of the curve where it becomes actually perpendicular, the value of $\frac{dy}{dx}$ will be infinitely great.

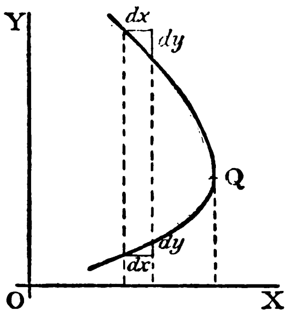

FIG. 18.

Now that we understand that $\frac{dy}{dx}$ measures the steepness of a curve at any point, let us turn to some of the equations which we have already learned how to differentiate.

(1) As the simplest case take this:

$$y = x + b.$$

It is plotted out in Fig. 19, using equal scales for x and y. If we put $x = 0$, then the corresponding ordinate will be $y = b$; that is to say, the "curve" crosses the y-axis at the height b. From here it ascends at 45°; for whatever values we give to x to the right, we have an equal y to ascend. The line has a gradient of 1 in 1.

Now differentiate $y=x+b$, by the rules we have already learned (pp. 22 and 26 *ante*), and we get $\frac{dy}{dx}=1$.

The slope of the line is such that for every little step dx to the right, we go an equal little step dy upward. And this slope is constant—always the same slope.

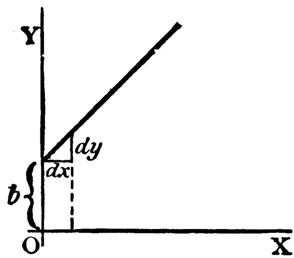

FIG. 19.

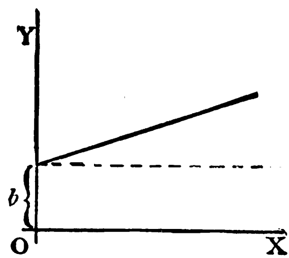

FIG. 20.

(2) Take another case:

$$y=ax+b.$$

We know that this curve, like the preceding one, will start from a height b on the y-axis. But before we draw the curve, let us find its slope by differentiating; which gives us $\frac{dy}{dx}=a$. The slope will be constant, at an angle, the tangent of which is here called a. Let us assign to a some numerical value—say $\frac{1}{3}$. Then we must give it such a slope that it ascends 1 in 3; or *dx* will be 3 times as great as *dy*; as magnified in Fig. 21. So, draw the line in Fig. 20 at this slope.

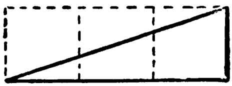

FIG. 21.

(3) Now for a slightly harder case.

Let $$y = ax^2 + b.$$

Again the curve will start on the y-axis at a height b above the origin.

Now differentiate. [If you have forgotten, turn back to p. 26; or, rather, *don't* turn back, but think out the differentiation.]

$$\frac{dy}{dx} = 2ax.$$

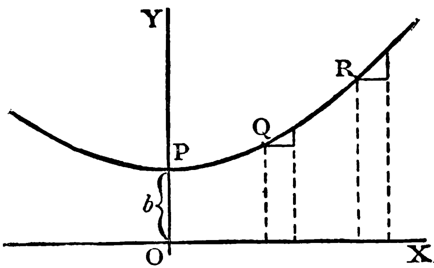

FIG. 22.

This shows that the steepness will not be constant: it increases as x increases. At the starting point *P*, where $x=0$, the curve (Fig. 22) has no steepness —that is, it is level. On the left of the origin, where x has negative values, $\frac{dy}{dx}$ will also have negative values, or will descend from left to right, as in the Figure.

Let us illustrate this by working out a particular instance. Taking the equation

$$y=\tfrac{1}{4}x^2+3,$$

and differentiating it, we get

$$\frac{dy}{dx}=\tfrac{1}{2}x.$$

Now assign a few successive values, say from 0 to 5, to x; and calculate the corresponding values of y by the first equation; and of $\frac{dy}{dx}$ from the second equation. Tabulating results, we have:

x	0	1	2	3	4	5
y	3	$3\frac{1}{4}$	4	$5\frac{1}{4}$	7	$9\frac{1}{4}$
$\frac{dy}{dx}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$

Then plot them out in two curves, Figs. 23 and 24 in Fig. 23 plotting the values of y against those of x, and in Fig. 24 those of $\frac{dy}{dx}$ against those of x. For any assigned value of x, the *height* of the ordinate in the second curve is proportional to the *slope* of the first curve.

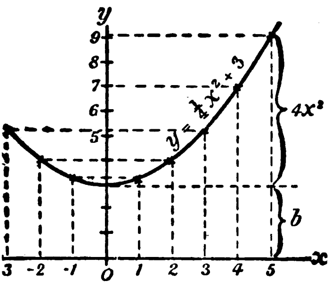

FIG. 23.

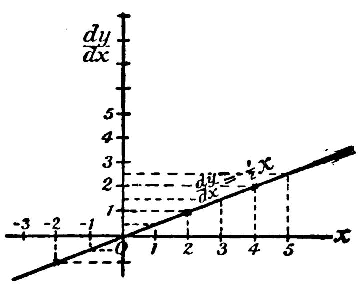

FIG. 24.

If a curve comes to a sudden cusp, as in Fig. 25, the slope at that point suddenly changes from a slope

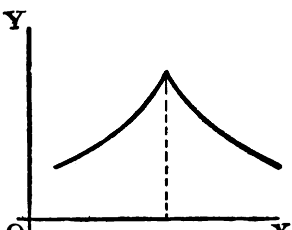

FIG. 25.

upward to a slope downward. In that case $\frac{dy}{dx}$ will clearly undergo an abrupt change from a positive to a negative value.

The following examples show further applications of the principles just explained.

(4) Find the slope of the tangent to the curve

$$y=\frac{1}{2x}+3,$$

at the point where $x=-1$. Find the angle which this tangent makes with the curve $y=2x^2+2$.

The slope of the tangent is the slope of the curve at the point where they touch one another (see p. 77); that is, it is the $\frac{dy}{dx}$ of the curve for that point. Here $\frac{dy}{dx}=-\frac{1}{2x^2}$ and for $x=-1$, $\frac{dy}{dx}=-\frac{1}{2}$, which is the slope of the tangent and of the curve at that point. The tangent, being a straight line, has for equation $y=ax+b$, and its slope is $\frac{dy}{dx}=a$, hence $a=-\frac{1}{2}$. Also if $x=-1$, $y=\frac{1}{2(-1)}+3=2\frac{1}{2}$; and as the tangent passes by this point, the coordinates of the point must satisfy the equation of the tangent, namely

$$y=-\frac{1}{2}x+b,$$

so that $2\frac{1}{2}=-\frac{1}{2}\times(-1)+b$ and $b=2$; the equation of the tangent is therefore $y=-\frac{1}{2}x+2$.

Now, when two curves meet, the intersection being a point common to both curves, its coordinates must satisfy the equation of each one of the two curves;

that is, it must be a solution of the system of simultaneous equations formed by coupling together the equations of the curves. Here the curves meet one another at points given by the solution of

$$\begin{cases} y=2x^2+2, \\ y=-\frac{1}{2}x+2 \end{cases} \text{ or } \quad 2x^2+2=-\tfrac{1}{2}x+2;$$

that is, $$x(2x+\tfrac{1}{2})=0.$$

This equation has for its solutions $x=0$ and $x=-\frac{1}{4}$. The slope of the curve $y=2x^2+2$ at any point is

$$\frac{dy}{dx}=4x.$$

For the point where $x=0$, this slope is zero; the curve is horizontal. For the point where

$$x=-\frac{1}{4},\quad \frac{dy}{dx}=-1;$$

hence the curve at that point slopes downwards to the right at such an angle θ with the horizontal that $\tan\theta=1$; that is, at 45° to the horizontal.

The slope of the straight line is $-\frac{1}{2}$; that is, it slopes downwards to the right and makes with the horizontal an angle ϕ such that $\tan\phi=\frac{1}{2}$; that is, an angle of 26° 34′. It follows that at the first point the curve cuts the straight line at an angle of 26° 34′, while at the second it cuts it at an angle of $45°-26°\,34'=18°\,26'$.

(5) A straight line is to be drawn, through a point whose coordinates are $x=2$, $y=-1$, as tangent to the curve $y=x^2-5x+6$. Find the coordinates of the point of contact.

The slope of the tangent must be the same as the $\frac{dy}{dx}$ of the curve; that is, $2x-5$.

The equation of the straight line is $y=ax+b$, and as it is satisfied for the values $x=2$, $y=-1$, then $-1=a\times 2+b$; also, its $\frac{dy}{dx}=a=2x-5$.

The x and the y of the point of contact must also satisfy both the equation of the tangent and the equation of the curve.

We have then

$$\left\{\begin{array}{rll} y &= x^2-5x+6, & \ldots\ldots\text{(i)}\\ y &= ax+b, & \ldots\ldots\text{(ii)}\\ -1 &= 2a+b, & \ldots\ldots\text{(iii)}\\ a &= 2x-5, & \ldots\ldots\text{(iv)}\end{array}\right.$$

four equations in a, b, x, y.

Equations (i) and (ii) give $x^2-5x+6=ax+b$.

Replacing a and b by their value in this, we get

$$x^2-5x+6=(2x-5)x-1-2(2x-5),$$

which simplifies to $x^2-4x+3=0$, the solutions of which are: $x=3$ and $x=1$. Replacing in (i), we get $y=0$ and $y=2$ respectively; the two points of contact are then $x=1$, $y=2$; and $x=3$, $y=0$.

Note.—In all exercises dealing with curves, students will find it extremely instructive to verify the deductions obtained by actually plotting the curves.

Exercises VIII. (See page 291 for Answers.)

(1) Plot the curve $y=\frac{3}{4}x^2-5$, using a scale of millimetres. Measure at points corresponding to different values of x, the angle of its slope.

Find, by differentiating the equation, the expression for slope; and see, from a Table of Natural Tangents, whether this agrees with the measured angle.

(2) Find what will be the slope of the curve

$$y=0{\cdot}12x^3-2,$$

at the particular point that has as abscissa $x=2$.

(3) If $y=(x-a)(x-b)$, show that at the particular point of the curve where $\frac{dy}{dx}=0$, x will have the value $\frac{1}{2}(a+b)$.

(4) Find the $\frac{dy}{dx}$ of the equation $y=x^3+3x$; and calculate the numerical values of $\frac{dy}{dx}$ for the points corresponding to $x=0$, $x=\frac{1}{2}$, $x=1$, $x=2$.

(5) In the curve to which the equation is $x^2+y^2=4$, find the values of x at those points where the slope $=1$.

(6) Find the slope, at any point, of the curve whose equation is $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$; and give the numerical value of the slope at the place where $x=0$, and at that where $x=1$.

(7) The equation of a tangent to the curve $y=5-2x+0{\cdot}5x^3$, being of the form $y=mx+n$, where m and n are constants, find the value of m and n if

the point where the tangent touches the curve has $x=2$ for abscissa.

(8) At what angle do the two curves

$$y=3{\cdot}5x^2+2 \quad \text{and} \quad y=x^2-5x+9{\cdot}5$$

cut one another?

(9) Tangents to the curve $y=\pm\sqrt{25-x^2}$ are drawn at points for which $x=3$ and $x=4$, the value of y being positive. Find the coordinates of the point of intersection of the tangents and their mutual inclination.

(10) A straight line $y=2x-b$ touches a curve $y=3x^2+2$ at one point. What are the coordinates of the point of contact, and what is the value of b?

CHAPTER XI.

MAXIMA AND MINIMA.

A QUANTITY which varies continuously is said to pass by (or through) a maximum or minimum value when, in the course of its variation, the immediately preceding and following values are *both* smaller or greater, respectively, than the value referred to. An infinitely great value is therefore not a maximum value.

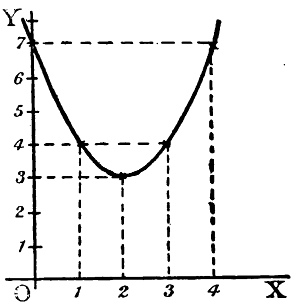

FIG. 26.

One of the principal uses of the process of differentiating is to find out under what conditions the value of the thing differentiated becomes a maximum, or a minimum. This is often exceedingly important in engineering questions, where it is most desirable to know what conditions will make the cost of working a minimum, or will make the efficiency a maximum.

Now, to begin with a concrete case, let us take the equation $$y = x^2 - 4x + 7.$$

By assigning a number of successive values to x, and finding the corresponding values of y, we can readily see that the equation represents a curve with a minimum.

x	0	1	2	3	4	5
y	7	4	3	4	7	12

These values are plotted in Fig. 26, which shows that y has apparently a minimum value of 3, when x is made equal to 2. But are you sure that the minimum occurs at 2, and not at $2\frac{1}{4}$ or at $1\frac{3}{4}$?

Of course it would be possible with any algebraic expression to work out a lot of values, and in this way arrive gradually at the particular value that may be a maximum or a minimum.

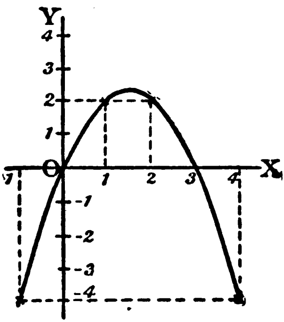

FIG. 27.

Here is another example:

Let $$y = 3x - x^2.$$

Calculate a few values thus:

x	−1	0	1	2	3	4	5
y	−4	0	2	2	0	−4	−10

Plot these values as in Fig. 27.

It will be evident that there will be a maximum somewhere between $x=1$ and $x=2$; and the thing *looks* as if the maximum value of y ought to be about $2\frac{1}{4}$. Try some intermediate values. If $x=1\frac{1}{4}$, $y=2{\cdot}187$; if $x=1\frac{1}{2}$, $y=2{\cdot}25$; if $x=1{\cdot}6$, $y=2{\cdot}24$. How can we be sure that $2{\cdot}25$ is the real maximum, or that it occurs exactly when $x=1\frac{1}{2}$?

Now it may sound like juggling to be assured that there is a way by which one can arrive straight at a maximum (or minimum) value without making a lot of preliminary trials or guesses. And that way depends on differentiating. Look back to an earlier page (81) for the remarks about Figs. 14 and 15, and you will see that whenever a curve gets either to its maximum or to its minimum height, at that point its $\frac{dy}{dx}=0$. Now this gives us the clue to the dodge that is wanted. When there is put before you an equation, and you want to find that value of x that will make its y a minimum (or a maximum), *first differentiate it*, and having done so, write its $\frac{dy}{dx}$ as *equal to zero*, and then solve for x. Put this particular value of x into the original equation, and you will then get the required value of y. This process is commonly called "equating to zero."

To see how simply it works, take the example with which this chapter opens, namely

$$y=x^2-4x+7.$$

Differentiating, we get:

$$\frac{dy}{dx}=2x-4.$$

Now equate this to zero, thus:

$$2x-4=0.$$

Solving this equation for x, we get:

$$2x=4,$$
$$x=2.$$

Now, we know that the maximum (or minimum) will occur exactly when $x=2$.

Putting the value $x=2$ into the original equation, we get

$$\begin{aligned} y&=2^2-(4\times 2)+7 \\ &=4-8+7 \\ &=3. \end{aligned}$$

Now look back at Fig 26, and you will see that the minimum occurs when $x=2$, and that this minimum of $y=3$.

Try the second example (Fig. 24), which is

$$y=3x-x^2.$$

Differentiating, $\frac{dy}{dx}=3-2x.$

Equating to zero,

$$3-2x=0,$$

whence $x=1\frac{1}{2}$;

and putting this value of x into the original equation, we find:

$$y=4\tfrac{1}{2}-(1\tfrac{1}{2}\times 1\tfrac{1}{2}),$$
$$y=2\tfrac{1}{4}.$$

This gives us exactly the information as to which the method of trying a lot of values left us uncertain.

Now, before we go on to any further cases, we have two remarks to make. When you are told to equate $\frac{dy}{dx}$ to zero, you feel at first (that is if you have any wits of your own) a kind of resentment, because you know that $\frac{dy}{dx}$ has all sorts of different values at different parts of the curve, according to whether it is sloping up or down. So, when you are suddenly told to write

$$\frac{dy}{dx}=0,$$

you resent it, and feel inclined to say that it can't be true. Now you will have to understand the essential difference between "an equation," and "an equation of condition." Ordinarily you are dealing with equations that are true in themselves; but, on occasions, of which the present are examples, you have to write down equations that are not necessarily true, but are only true if certain conditions are to be fulfilled; and you write them down in order, by solving them, to find the conditions which make them true Now we want to find the particular value that x has when the curve is neither sloping up nor sloping down, that is, at the particular place where $\frac{dy}{dx}=0$. So, writing $\frac{dy}{dx}=0$ does *not* mean that it always is $=0$; but you write it down *as a condition* in order to see how much x will come out if $\frac{dy}{dx}$ is to be zero.

The second remark is one which (if you have any wits of your own) you will probably have already made: namely, that this much-belauded process of equating to zero entirely fails to tell you whether the x that you thereby find is going to give you a *maximum* value of y or a *minimum* value of y. Quite so. It does not of itself discriminate; it finds for you the right value of x but leaves you to find out for yourselves whether the corresponding y is a maximum or a minimum. Of course, if you have plotted the curve, you know already which it will be.

For instance, take the equation:

$$y = 4x + \frac{1}{x}.$$

Without stopping to think what curve it corresponds to, differentiate it, and equate to zero:

$$\frac{dy}{dx} = 4 - x^{-2} = 4 - \frac{1}{x^2} = 0;$$

whence $x = \frac{1}{2}$;

and, inserting this value,

$$y = 4$$

will be either a maximum or else a minimum. But which? You will hereafter be told a way, depending upon a second differentiation, (see Chap. XII., p. 112). But at present it is enough if you will simply try any other value of x differing a little from the one found, and see whether with this altered value the corresponding value of y is less or greater than that already found.

Try another simple problem in maxima and minima. Suppose you were asked to divide any number into two parts, such that the product was a maximum? How would you set about it if you did not know the trick of equating to zero? I suppose you could worry it out by the rule of try, try, try again. Let 60 be the number. You can try cutting it into two parts, and multiplying them together. Thus, 50 times 10 is 500; 52 times 8 is 416; 40 times 20 is 800; 45 times 15 is 675; 30 times 30 is 900. This looks like a maximum: try varying it. 31 times 29 is 899, which is not so good; and 32 times 28 is 896, which is worse. So it seems that the biggest product will be got by dividing into two equal halves.

Now see what the calculus tells you. Let the number to be cut into two parts be called n. Then if x is one part, the other will be $n-x$, and the product will be $x(n-x)$ or $nx-x^2$. So we write $y=nx-x^2$. Now differentiate and equate to zero;

$$\frac{dy}{dx}=n-2x=0.$$

Solving for x, we get $\frac{n}{2}=x$.

So now we *know* that whatever number n may be, we must divide it into two equal parts if the product of the parts is to be a maximum; and the value of that maximum product will always be $=\frac{1}{4}n^2$.

This is a very useful rule, and applies to any number of factors, so that if $m+n+p=$ a constant number, $m\times n\times p$ is a maximum when $m=n=p$.

Test Case.

Let us at once apply our knowledge to a case that we can test.

Let $$y = x^2 - x;$$

and let us find whether this function has a maximum or minimum; and if so, test whether it is a maximum or a minimum.

Differentiating, we get

$$\frac{dy}{dx} = 2x - 1.$$

Equating to zero, we get

$$2x - 1 = 0,$$

whence $$2x = 1,$$

or $$x = \tfrac{1}{2}.$$

That is to say, when x is made $= \frac{1}{2}$, the corresponding value of y will be either a maximum or a minimum. Accordingly, putting $x = \frac{1}{2}$ in the original equation, we get

$$y = (\tfrac{1}{2})^2 - \tfrac{1}{2},$$

or $$y = -\tfrac{1}{4}.$$

Is this a maximum or a minimum? To test it, try putting x a little bigger than $\frac{1}{2}$,—say make $x = 0{\cdot}6$ Then $$y = (0{\cdot}6)^2 - 0{\cdot}6 = 0{\cdot}36 - 0{\cdot}6 = -0{\cdot}24,$$

which is higher up than $-0{\cdot}25$; showing that $y = -0{\cdot}25$ is a *minimum*.

Plot the curve for yourself, and verify the calculation.

Further Examples.

A most interesting example is afforded by a curve that has both a maximum and a minimum. Its equation is:

$$y=\tfrac{1}{3}x^3-2x^2+3x+1.$$

Now

$$\frac{dy}{dx}=x^2-4x+3.$$

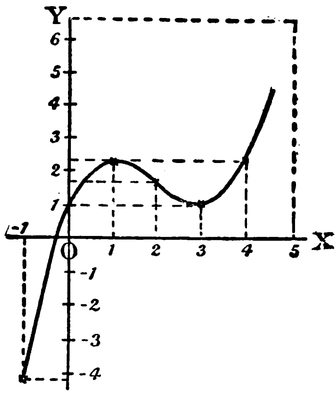

FIG. 28.

Equating to zero, we get the quadratic,

$$x^2-4x+3=0;$$

and solving the quadratic gives us *two* roots, viz.

$$\begin{cases} x=3 \\ x=1. \end{cases}$$

Now, when $x=3$, $y=1$; and when $x=1$, $y=2\tfrac{1}{3}$. The first of these is a minimum, the second a maximum.

The curve itself may be plotted (as in Fig. 28) from the values calculated, as below, from the original equation.

x	-1	0	1	2	3	4	5	6
y	$-4\frac{1}{3}$	1	$2\frac{1}{3}$	$1\frac{2}{3}$	1	$2\frac{1}{3}$	$7\frac{2}{3}$	19

A further exercise in maxima and minima is afforded by the following example:

The equation to a circle of radius r, having its centre C at the point whose coordinates are $x=a$, $y=b$, as depicted in Fig. 29, is:

$$(y-b)^2+(x-a)^2=r^2.$$

This may be transformed into

$$y=\sqrt{r^2-(x-a)^2}+b.$$

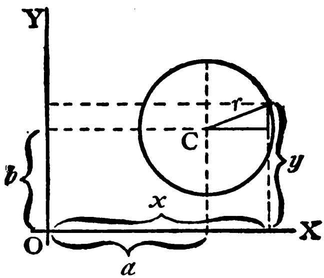

FIG. 29.

Now we know beforehand, by mere inspection of the figure, that when $x=a$, y will be either at its maximum value, $b+r$, or else at its minimum value, $b-r$. But let us not take advantage of this knowledge; let us set about finding what value of x will make y a maximum or a minimum, by the process of differentiating and equating to zero.

$$\frac{dy}{dx}=\frac{1}{2}\frac{1}{\sqrt{r^2-(x-a)^2}}\times(2a-2x),$$

which reduces to

$$\frac{dy}{dx}=\frac{a-x}{\sqrt{r^2-(x-a)^2}}.$$

Then the condition for y being maximum or minimum is:

$$\frac{a-x}{\sqrt{r^2-(x-a)^2}}=0.$$

Since no value whatever of x will make the denominator infinite, the only condition to give zero is

$$x=a.$$

Inserting this value in the original equation for the circle, we find

$$y=\sqrt{r^2}+b;$$

and as the root of r^2 is either $+r$ or $-r$, we have two resulting values of y,

$$\begin{cases} y=b+r \\ y=b-r \end{cases}$$

The first of these is the maximum, at the top; the second the minimum, at the bottom.

If the curve is such that there is no place that is a maximum or minimum, the process of equating to zero will yield an impossible result. For instance:

Let $$y = ax^3 + bx + c.$$

Then $$\frac{dy}{dx} = 3ax^2 + b.$$

Equating this to zero, we get $3ax^2 + b = 0$, $x^2 = \frac{-b}{3a}$, and $x = \sqrt{\frac{-b}{3a}}$, which is impossible, supposing a and b to have the same sign.

Therefore y has no maximum nor minimum.

A few more worked examples will enable you to thoroughly master this most interesting and useful application of the calculus.

(1) What are the sides of the rectangle of maximum area inscribed in a circle of radius R?

If one side be called x,

$$\text{the other side} = \sqrt{(\text{diagonal})^2 - x^2};$$

and as the diagonal of the rectangle is necessarily a diameter, the other side $= \sqrt{4R^2 - x^2}$.

Then, area of rectangle $S = x\sqrt{4R^2 - x^2}$,

$$\frac{dS}{dx} = x \times \frac{d(\sqrt{4R^2 - x^2})}{dx} + \sqrt{4R^2 - x^2} \times \frac{d(x)}{dx}.$$

If you have forgotten how to differentiate $\sqrt{4R^2 - x^2}$, here is a hint: write $4R^2 - x^2 = w$ and $y = \sqrt{w}$, and seek $\frac{dy}{dw}$ and $\frac{dw}{dx}$; fight it out, and only if you can't get on refer to page 67.

You will get

$$\frac{dS}{dx}=x\times-\frac{x}{\sqrt{4R^2-x^2}}+\sqrt{4R^2-x^2}=\frac{4R^2-2x^2}{\sqrt{4R^2-x^2}}.$$

For maximum or minimum we must have

$$\frac{4R^2-2x^2}{\sqrt{4R^2-x^2}}=0,$$

that is, $4R^2-2x^2=0$ and $x=R\sqrt{2}$.

The other side $=\sqrt{4R^2-2R^2}=R\sqrt{2}$; the two sides are equal; the figure is a square the side of which is equal to the diagonal of the square constructed on the radius. In this case it is, of course, a maximum with which we are dealing.

(2) What is the radius of the opening of a conical vessel the sloping side of which has a length l when the capacity of the vessel is greatest?

If R be the radius and H the corresponding height, $H=\sqrt{l^2-R^2}$.

Volume $V=\pi R^2\times\frac{H}{3}=\pi R^2\times\frac{\sqrt{l^2-R^2}}{3}$.

Proceeding as in the previous problem, we get

$$\frac{dV}{dR}=\pi R^2\times-\frac{R}{3\sqrt{l^2-R^2}}+\frac{2\pi R}{3}\sqrt{l^2-R^2}$$

$$=\frac{2\pi R(l^2-R^2)-\pi R^3}{3\sqrt{l^2-R^2}}=0$$

for maximum or minimum.

Or, $2\pi R(l^2-R^2)-\pi R^3=0$, and $R=l\sqrt{\frac{2}{3}}$, for a maximum, obviously.

(3) Find the maxima and minima of the function

$$y=\frac{x}{4-x}+\frac{4-x}{x}.$$

We get

$$\frac{dy}{dx}=\frac{(4-x)-(-x)}{(4-x)^2}+\frac{-x-(4-x)}{x^2}=0$$

for maximum or minimum; or

$$\frac{4}{(4-x)^2}-\frac{4}{x^2}=0 \quad \text{and} \quad x=2.$$

There is only one value, hence only one maximum or minimum.

For $x=2$, $y=2$,

for $x=1{\cdot}5$, $y=2{\cdot}27$,

for $x=2{\cdot}5$, $y=2{\cdot}27$;

it is therefore a minimum. (It is instructive to plot the graph of the function.)

(4) Find the maxima and minima of the function $y=\sqrt{1+x}+\sqrt{1-x}$. (It will be found instructive to plot the graph.)

Differentiating gives at once (see example No. 1, p. 68)

$$\frac{dy}{dx}=\frac{1}{2\sqrt{1+x}}-\frac{1}{2\sqrt{1-x}}=0$$

for maximum or minimum.

Hence $\sqrt{1+x}=\sqrt{1-x}$ and $x=0$, the only solution.

For $x=0$, $y=2$.

For $x=\pm 0{\cdot}5$, $y=1{\cdot}932$, so this is a maximum.

(5) Find the maxima and minima of the function

$$y=\frac{x^2-5}{2x-4}.$$

We have

$$\frac{dy}{dx}=\frac{(2x-4)\times 2x-(x^2-5)2}{(2x-4)^2}=0$$

for maximum or minimum; or

$$\frac{2x^2-8x+10}{(2x-4)^2}=0;$$

or $x^2-4x+5=0$; which has for solutions

$$x=\tfrac{5}{2}\pm\sqrt{-1}.$$

These being imaginary, there is no real value of x for which $\frac{dy}{dx}=0$; hence there is neither maximum nor minimum.

(6) Find the maxima and minima of the function

$$(y-x^2)^2=x^5.$$

This may be written $y=x^2\pm x^{\frac{5}{2}}$.

$\frac{dy}{dx}=2x\pm\frac{5}{2}x^{\frac{3}{2}}=0$ for maximum or minimum;

that is, $x(2\pm\frac{5}{2}x^{\frac{1}{2}})=0$, which is satisfied for $x=0$, and for $2\pm\frac{5}{2}x^{\frac{1}{2}}=0$, that is for $x=\frac{16}{25}$. So there are two solutions.

Taking first $x=0$. If $x=-0{\cdot}5$, $y=0{\cdot}25\pm\sqrt[2]{-({\cdot}5)^5}$, and if $x=+0{\cdot}5$, $y=0{\cdot}25\pm\sqrt[2]{({\cdot}5)^5}$. On one side y is imaginary; that is, there is no value of y that can be represented by a graph; the latter is therefore entirely on the right side of the axis of y (see Fig. 30).

On plotting the graph it will be found that the curve goes to the origin, as if there were a minimum there; but instead of continuing beyond, as it should do for a minimum, it retraces its steps (forming what is called a "cusp"). There is no minimum, therefore, although the condition for a minimum is satisfied, namely $\frac{dy}{dx}=0$. It is necessary therefore always to check by taking one value on either side.

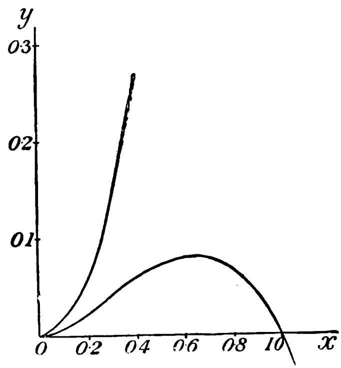

FIG. 30.

Now, if we take $x=\frac{16}{25}=0{\cdot}64$. If $x=0{\cdot}64$, $y=0{\cdot}7373$ and $y=0{\cdot}0819$; if $x=0{\cdot}6$, y becomes $0{\cdot}6389$ and $0{\cdot}0811$; and if $x=0{\cdot}7$, y becomes $0{\cdot}8996$ and $0{\cdot}0804$.

This shows that there are two branches of the curve, the upper one does not pass through a maximum, but the lower one does.

(7) A cylinder whose height is twice the radius of the base is increasing in volume, so that all its parts

keep always in the same proportion to each other; that is, at any instant, the cylinder is *similar* to the original cylinder. When the radius of the base is r feet, the surface area is increasing at the rate of 20 square inches per second; at what rate per second is its volume then increasing?

$$\text{Area} = S = 2(\pi r^2) + 2\pi r \times 2r = 6\pi r^2.$$

$$\text{Volume} = V = \pi r^2 \times 2r = 2\pi r^3.$$

$$\frac{dS}{dt} = 12\pi r \frac{dr}{dt} = 20; \quad \frac{dr}{dt} = \frac{20}{12\pi r};$$

$$\frac{dV}{dt} = 6\pi r^2 \frac{dr}{dt}; \quad \text{and}$$

$$\frac{dV}{dt} = 6\pi r^2 \times \frac{20}{12\pi r} = 10r.$$

The volume changes at the rate of $10r$ cubic inches per second.

Make other examples for yourself. There are few subjects which offer such a wealth for interesting examples.

Exercises IX. (See page 292 for Answers.)

(1) What values of x will make y a maximum and a minimum, if $y = \frac{x^2}{x+1}$?

(2) What value of x will make y a maximum in the equation $y = \frac{x}{a^2 + x^2}$?

(3) A line of length p is to be cut up into 4 parts and put together as a rectangle. Show that the area of the rectangle will be a maximum if each of its sides is equal to $\frac{1}{4}p$.

(4) A piece of string 30 inches long has its two ends joined together and is stretched by 3 pegs so as to form a triangle. What is the largest triangular area that can be enclosed by the string?
(*Hint*: Apply last three lines of p. 99.)

(5) Plot the curve corresponding to the equation

$$y=\frac{10}{x}+\frac{10}{8-x};$$

also find $\frac{dy}{dx}$, and deduce the value of x that will make y a minimum; and find that minimum value of y.

(6) If $y=x^5-5x$, find what values of x will make y a maximum or a minimum.

(7) What is the smallest square that can be inscribed in a given square?

(8) Inscribe in a given cone, the height of which is equal to the radius of the base, a cylinder (*a*) whose volume is a maximum; (*b*) whose lateral area is a maximum; (*c*) whose total area is a maximum.

(9) Inscribe in a sphere, a cylinder (*a*) whose volume is a maximum; (*b*) whose lateral area is a maximum; (*c*) whose total area is a maximum.

(10) A spherical balloon is increasing in volume. If, when its radius is r feet, its volume is increasing at the rate of 4 cubic feet per second, at what rate is its surface then increasing?

(11) Inscribe in a given sphere a cone whose volume is a maximum.

(12) The current C given by a battery of N similar voltaic cells is $C=\dfrac{n\times E}{R+\dfrac{rn^2}{N}}$, where E, R, r, are constants and n is the number of cells coupled in series. Find the proportion of n to N for which the current is greatest.

CHAPTER XII.

CURVATURE OF CURVES.

RETURNING to the process of successive differentiation, it may be asked: Why does anybody want to differentiate twice over? We know that when the variable quantities are space and time, by differentiating twice over we get the acceleration of a moving body, and that in the geometrical interpreta-

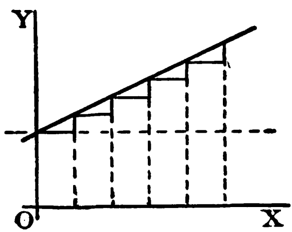

FIG. 31.

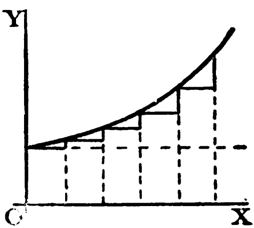

FIG. 32.

tion, as applied to curves, $\dfrac{dy}{dx}$ means the *slope* of the curve. But what can $\dfrac{d^2y}{dx^2}$ mean in this case? Clearly it means the rate (per unit of length x) at which the slope is changing—in brief, it is *an indication of the manner in which the slope of the portion of curve considered varies*, that is, whether the slope of the curve increases or decreases when x increases, or, in other words, whether the curve curves up or down towards the right

Suppose a slope constant, as in Fig. 31.

Here, $\frac{dy}{dx}$ is of constant value.

Suppose, however, a case in which, like Fig. 32, the slope itself is getting greater upwards; then $\frac{d\left(\frac{dy}{dx}\right)}{dx}$, that is, $\frac{d^2y}{dx^2}$, will be *positive.*

If the slope is becoming less as you go to the right (as in Fig. 14, p. 81), or as in Fig. 33, then, even though the curve may be going upward, since the change is such as to diminish its slope, its $\frac{d^2y}{dx^2}$ will be *negative.*

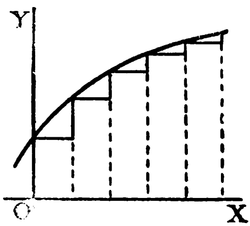

FIG. 33.

It is now time to initiate you into another secret—how to tell whether the result that you get by "equating to zero" is a maximum or a minimum. The trick is this: After you have differentiated (so as to get the expression which you equate to zero), you then differentiate a second time, and look whether the result of the second differentiation is *positive* or *negative.* If $\frac{d^2y}{dx^2}$ comes out *positive*, then you know that the value of y which you got was a *minimum*; but if $\frac{d^2y}{dx^2}$ comes out *negative*, then the value of y which you got must be a *maximum*. That's the rule.

The reason of it ought to be quite evident. Think of any curve that has a minimum point in it, like Fig. 15 (p. 81), or like Fig. 34, where the point of minimum y is marked M, and the curve is *concave* upwards. To the left of M the slope is downward, that is, negative, and is getting less negative. To the right of M the slope has become upward, and is

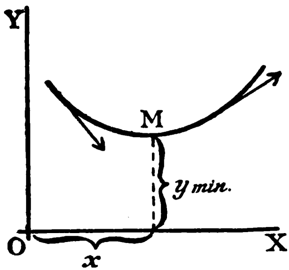

FIG. 34.

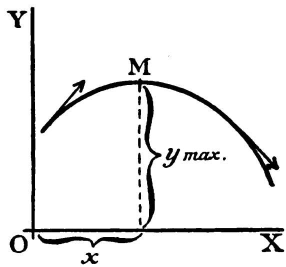

FIG. 35.

getting more and more upward. Clearly the change of slope as the curve passes through M is such that $\frac{d^2y}{dx^2}$ is *positive*, for its operation, as x increases toward the right, is to convert a downward slope into an upward one.

Similarly, consider any curve that has a maximum point in it, like Fig. 16 (p. 82), or like Fig. 35, where the curve is *convex*, and the maximum point is marked M. In this case, as the curve passes through M from left to right, its upward slope is converted into a downward or negative slope, so that in this case the "slope of the slope" $\dfrac{d^2y}{dx^2}$ is *negative*.

Go back now to the examples of the last chapter and verify in this way the conclusions arrived at as to whether in any particular case there is a maximum or a minimum. You will find below a few worked out examples.

---

(1) Find the maximum or minimum of

$(a)\ y=4x^2-9x-6; \qquad (b)\ y=6+9x-4x^2;$

and ascertain if it be a maximum or a minimum in each case.

$(a)\ \dfrac{dy}{dx}=8x-9=0;\ x=1\tfrac{1}{8};$ and $y=-11{\cdot}065.$

$\dfrac{d^2y}{dx^2}=8;$ it is $+$; hence it is a minimum.

$(b)\ \dfrac{dy}{dx}=9-8x=0;\ x=1\tfrac{1}{8};$ and $y=+11{\cdot}065.$

$\dfrac{d^2y}{dx^2}=-8;$ it is $-$; hence it is a maximum.

(2) Find the maxima and minima of the function $y=x^3-3x+16$.

$\dfrac{dy}{dx}=3x^2-3=0;\ x^2=1;$ and $x=\pm1.$

$\dfrac{d^2y}{dx^2}=6x;$ for $x=1$; it is $+$;

hence $x=1$ corresponds to a minimum $y=14$. For $x=-1$ it is $-$; hence $x=-1$ corresponds to a maximum $y=+18$.

(3) Find the maxima and minima of $y=\dfrac{x-1}{x^2+2}$.

$$\frac{dy}{dx}=\frac{(x^2+2)\times 1-(x-1)\times 2x}{(x^2+2)^2}=\frac{2x-x^2+2}{(x^2+2)^2}=0;$$

or $x^2-2x-2=0$, whose solutions are $x=+2{\cdot}73$ and $x=-0{\cdot}73$.

$$\frac{d^2y}{dx^2}=-\frac{(x^2+2)^2\times(2x-2)-(x^2-2x-2)(4x^3+8x)}{(x^2+2)^4}$$

$$=\frac{2x^5-6x^4-8x^3-8x^2-24x+8}{(x^2+2)^4}.$$

The denominator is always positive, so it is sufficient to ascertain the sign of the numerator.

If we put $x=2{\cdot}73$, the numerator is negative; the maximum, $y=0{\cdot}183$.

If we put $x=-0{\cdot}73$, the numerator is positive; the minimum, $y=-0{\cdot}683$.

(4) The expense C of handling the products of a certain factory varies with the weekly output P according to the relation $C=aP+\dfrac{b}{c+P}+d$, where a, b, c, d are positive constants. For what output will the expense be least?

$\dfrac{dC}{dP}=a-\dfrac{b}{(c+P)^2}=0$ for maximum or minimum;

hence $a=\dfrac{b}{(c+P)^2}$ and $P=\pm\sqrt{\dfrac{b}{a}}-c$.

As the output cannot be negative, $P=+\sqrt{\dfrac{b}{a}}-c$.

Now $$\frac{d^2C}{dP^2}=+\frac{b(2c+2P)}{(c+P)^4},$$

which is positive for all the values of P; hence $P=+\sqrt{\frac{b}{a}}-c$ corresponds to a minimum.

(5) The total cost per hour C of lighting a building with N lamps of a certain kind is

$$C=N\left(\frac{C_l}{t}+\frac{EPC_e}{1000}\right),$$

where E is the commercial efficiency (watts per candle),

P is the candle power of each lamp,

t is the average life of each lamp in hours,

C_l = cost of renewal in pence per hour of use,

C_e = cost of energy per 1000 watts per hour.

Moreover, the relation connecting the average life of a lamp with the commercial efficiency at which it is run is approximately $t=mE^n$, where m and n are constants depending on the kind of lamp.

Find the commercial efficiency for which the total cost of lighting will be least.

We have $$C=N\left(\frac{C_l}{m}E^{-n}+\frac{PC_e}{1000}E\right),$$

$$\frac{dC}{dE}=N\left(\frac{PC_e}{1000}-\frac{nC_l}{m}E^{-(n+1)}\right)=0$$

for maximum or minimum.

$$E^{n+1}=\frac{1000\times nC_l}{mPC_e} \quad\text{and}\quad E=\sqrt[n+1]{\frac{1000\times nC_l}{mPC_e}}.$$

This is clearly for minimum, since

$$\frac{d^2C}{dE^2}=N\left[(n+1)\frac{nC_l}{m}E^{-(n+2)}\right],$$

which is positive for a positive value of E.

For a particular type of 16 candle-power lamps, $C_l=17$ pence, $C_e=5$ pence; and it was found that $m=10$ and $n=3{\cdot}6$.

$$E=\sqrt[4\cdot6]{\frac{1000\times3{\cdot}6\times17}{10\times16\times5}}=2{\cdot}6 \text{ watts per candle-power.}$$

Exercises X. (You are advised to plot the graph of any numerical example.) (See p. 292 for the Answers.)

(1) Find the maxima and minima of

$$y=x^3+x^2-10x+8.$$

(2) Given $y=\frac{b}{a}x-cx^2$, find expressions for $\frac{dy}{dx}$, and for $\frac{d^2y}{dx^2}$; also find the value of x which makes y a maximum or a minimum, and show whether it is maximum or minimum.

(3) Find how many maxima and how many minima there are in the curve, the equation to which is

$$y=1-\frac{x^2}{2}+\frac{x^4}{24};$$

and how many in that of which the equation is

$$y=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}.$$

(4) Find the maxima and minima of

$$y=2x+1+\frac{5}{x^2}.$$

(5) Find the maxima and minima of

$$y=\frac{3}{x^2+x+1}.$$

(6) Find the maxima and minima of

$$y=\frac{5x}{2+x^2}.$$

(7) Find the maxima and minima of

$$y=\frac{3x}{x^2-3}+\frac{x}{2}+5.$$

(8) Divide a number N into two parts in such a way that three times the square of one part plus twice the square or the other part shall be a minimum.

(9) The efficiency u of an electric generator at different values of output x is expressed by the general equation:

$$u=\frac{x}{a+bx+cx^2};$$

where a is a constant depending chiefly on the energy losses in the iron and c a constant depending chiefly on the resistance of the copper parts. Find an expression for that value of the output at which the efficiency will be a maximum.

(10) Suppose it to be known that consumption of coal by a certain steamer may be represented by the formula $y=0{\cdot}3+0{\cdot}001v^3$; where y is the number of tons of coal burned per hour and v is the speed expressed in nautical miles per hour. The cost of wages, interest on capital, and depreciation of that ship are together equal, per hour, to the cost of 1 ton of coal. What speed will make the total cost of a voyage of 1000 nautical miles a minimum? And, if coal costs 10 shillings per ton, what will that minimum cost of the voyage amount to?

(11) Find the maxima and minima of

$$y = \pm\frac{x}{6}\sqrt{x(10-x)}.$$

(12) Find the maxima and minima of

$$y = 4x^3 - x^2 - 2x + 1$$

CHAPTER XIII.

OTHER USEFUL DODGES.

Partial Fractions.

WE have seen that when we differentiate a fraction we have to perform a rather complicated operation; and, if the fraction is not itself a simple one, the result is bound to be a complicated expression. If we could split the fraction into two or more simpler fractions such that their sum is equivalent to the original fraction, we could then proceed by differentiating each of these simpler expressions. And the result of differentiating would be the sum of two (or more) differentials, each one of which is relatively simple; while the final expression, though of course it will be the same as that which could be obtained without resorting to this dodge, is thus obtained with much less effort and appears in a simplified form.

Let us see how to reach this result. Try first the job of adding two fractions together to form a resultant fraction. Take, for example, the two fractions $\frac{1}{x+1}$ and $\frac{2}{x-1}$. Every schoolboy can add these together and find their sum to be $\frac{3x+1}{x^2-1}$. And in the same way he can add together three or more fractions. Now this process can certainly be reversed: that is to say that, if this last expression were given, it is certain that it can somehow be split back again into its original components or partial fractions. Only we do not know in every case that may be presented to us *how* we can so split it. In order to find this out we shall consider a simple case at first. But it is important to bear in mind that all which follows applies only to what are called "proper" algebraic fractions, meaning fractions like the above, which have the numerator of *a lesser degree* than the denominator; that is, those in which the highest index of x is less in the numerator than in the denominator. If we have to deal with such an expression as $\frac{x^2+2}{x^2-1}$, we can simplify it by division, since it is equivalent to $1+\frac{3}{x^2-1}$; and $\frac{3}{x^2-1}$ is a proper algebraic fraction to which the operation of splitting into partial fractions can be applied, as explained hereafter.

Case I. If we perform many additions of two or more fractions the denominators of which contain only terms in x, and no terms in x^2, x^3, or any other powers of x, we *always* find that *the denominator of the final resulting fraction is the product of the denominators* of the fractions which were added to form the result. It follows that by factorizing the denominator of this final fraction, we can find every one of the denominators of the partial fractions of which we are in search.

Suppose we wish to go back from $\frac{3x+1}{x^2-1}$ to the components which we know are $\frac{1}{x+1}$ and $\frac{2}{x-1}$. If we did not know what those components were we can still prepare the way by writing:

$$\frac{3x+1}{x^2-1}=\frac{3x+1}{(x+1)(x-1)}=\frac{\quad}{x+1}+\frac{\quad}{x-1},$$

leaving blank the places for the numerators until we know what to put there. We always may assume the sign between the partial fractions to be *plus*, since, if it be *minus*, we shall simply find the corresponding numerator to be negative. Now, since the partial fractions are *proper* fractions, the numerators are mere numbers without x at all, and we can call them A, B, C ... as we please. So, in this case, we have:

$$\frac{3x+1}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1}.$$

If, now, we perform the addition of these two partial fractions, we get $\frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$; and this must be equal to $\frac{3x+1}{(x+1)(x-1)}$. And, as the denominators in these two expressions are the same, the numerators must be equal, giving us:

$$3x+1=A(x-1)+B(x+1).$$

Now, this is an equation with two unknown quantities, and it would seem that we need another equation before we can solve them and find A and B.

But there is another way out of this difficulty. The equation must be true for all values of x; therefore it must be true for such values of x as will cause $x-1$ and $x+1$ to become zero, that is for $x=1$ and for $x=-1$ respectively. If we make $x=1$, we get $4=(A\times 0)+(B\times 2)$, so that $B=2$; and if we make $x=-1$, we get $-2=(A\times -2)+(B\times 0)$, so that $A=1$. Replacing the A and B of the partial fractions by these new values, we find them to become $\dfrac{1}{x+1}$ and $\dfrac{2}{x-1}$; and the thing is done.

As a further example, let us take the fraction $\dfrac{4x^2+2x-14}{x^3+3x^2-x-3}$. The denominator becomes zero when x is given the value 1; hence $x-1$ is a factor of it, and obviously then the other factor will be x^2+4x+3; and this can again be decomposed into $(x+1)(x+3)$. So we may write the fraction thus:

$$\frac{4x^2+2x-14}{x^3+3x^2-x-3}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x+3},$$

making three partial factors.

Proceeding as before, we find

$$4x^2+2x-14=A(x-1)(x+3)+B(x+1)(x+3)+C(x+1)(x-1).$$

Now, if we make $x=1$, we get:

$-8=(A\times 0)+B(2\times 4)+(C\times 0)$; that is, $B=-1$.

If $x=-1$, we get

$-12=A(-2\times 2)+(B\times 0)+(C\times 0)$; whence $A=3$.

If $x=-3$, we get:

$16=(A\times 0)+(B\times 0)+C(-2\times -4)$; whence $C=2$.

So then the partial fractions are:

$$\frac{3}{x+1}-\frac{1}{x-1}+\frac{2}{x+3},$$

which is far easier to differentiate with respect to x than the complicated expression from which it is derived.

Case II. If some of the factors of the denominator contain terms in x^2, and are not conveniently put into factors, then the corresponding numerator may contain a term in x, as well as a simple number, and hence it becomes necessary to represent this unknown numerator not by the symbol A but by $Ax+B$; the rest of the calculation being made as before.

Try, for instance: $\dfrac{-x^2-3}{(x^2+1)(x+1)}$.

$$\frac{-x^2-3}{(x^2+1)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1};$$

$$-x^2-3=(Ax+B)(x+1)+C(x^2+1).$$

Putting $x=-1$, we get $-4=C\times 2$; and $C=-2$;

hence $\quad -x^2-3=(Ax+B)(x+1)-2x^2-2$;

and $\quad x^2-1=Ax(x+1)+B(x+1)$.

Putting $x=0$, we get $-1=B$;

hence

$x^2-1=Ax(x+1)-x-1$; or $x^2+x=Ax(x+1)$;

and $\quad x+1=A(x+1)$.

so that $A=1$, and the partial fractions are:

$$\frac{x-1}{x^2+1}-\frac{2}{x+1}.$$

Take as another example the fraction

$$\frac{x^3-2}{(x^2+1)(x^2+2)}.$$

We get

$$\frac{x^3-2}{(x^2+1)(x^2+2)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2}$$
$$=\frac{(Ax+B)(x^2+2)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+2)}.$$

In this case the determination of A, B, C, D is not so easy. It will be simpler to proceed as follows: Since the given fraction and the fraction found by adding the partial fractions are equal, and have *identical* denominators, the numerators must also be identically the same. In such a case, and for such algebraical expressions as those with which we are dealing here, *the coefficients of the same powers of x are equal and of same sign.*

Hence, since

$$x^3-2=(Ax+B)(x^2+2)+(Cx+D)(x^2+1)$$
$$=(A+C)x^3+(B+D)x^2+(2A+C)x+2B+D,$$

we have $1=A+C$; $0=B+D$ (the coefficient of x^2 in the left expression being zero); $0=2A+C$; and $-2=2B+D$. Here are four equations, from which we readily obtain $A=-1$; $B=-2$; $C=2$; $D=2$; so that the partial fractions are $\dfrac{2(x+1)}{x^2+2}-\dfrac{x+2}{x^2+1}$

This method can always be used; but the method shown first will be found the quickest in the case of factors in x only.

Case III. When among the factors of the denominator there are some which are raised to some power, one must allow for the possible existence of partial fractions having for denominator the several powers of that factor up to the highest. For instance, in splitting the fraction $\frac{3x^2-2x+1}{(x+1)^2(x-2)}$ we must allow for the possible existence of a denominator $x+1$ as well as $(x+1)^2$ and $(x-2)$.

It may be thought, however, that, since the numerator of the fraction the denominator of which is $(x+1)^2$ may contain terms in x, we must allow for this in writing $Ax+B$ for its numerator, so that

$$\frac{3x^2-2x+1}{(x+1)^2(x-2)}=\frac{Ax+B}{(x+1)^2}+\frac{C}{x+1}+\frac{D}{x-2}.$$

If, however, we try to find A, B, C and D in this case, we fail, because we get four unknowns; and we have only three relations connecting them, yet

$$\frac{3x^2-2x+1}{(x+1)^2(x-2)}=\frac{x-1}{(x+1)^2}+\frac{1}{x+1}+\frac{1}{x-2}.$$

But if we write

$$\frac{3x^2-2x+1}{(x+1)^2(x-2)}=\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{C}{x-2},$$

we get

$$3x^2-2x+1=A(x-2)+B(x+1)(x-2)+C(x+1)^2,$$

which gives $C=1$ for $x=2$. Replacing C by its value, transposing, gathering like terms, and dividing by $x-2$, we get $-2x=A+B(x+1)$, which gives $A=-2$ for $x=-1$. Replacing A by its value, we get

$$2x=-2+B(x+1).$$

Hence $B=2$; so that the partial fractions are:

$$\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{x-2},$$

instead of $\frac{1}{x+1}+\frac{x-1}{(x+1)^2}+\frac{1}{x-2}$ stated above as being the fractions from which $\frac{3x^2-2x+1}{(x+1)^2(x-2)}$ was obtained. The mystery is cleared if we observe that $\frac{x-1}{(x+1)^2}$ can itself be split into the two fractions $\frac{1}{x+1}-\frac{2}{(x+1)^2}$, so that the three fractions given are really equivalent to

$$\frac{1}{x+1}+\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{x-2}=\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{x-2},$$

which are the partial fractions obtained.

We see that it is sufficient to allow for one numerical term in each numerator, and that we always get the ultimate partial fractions.

When there is a power of a factor of x^2 in the denominator, however, the corresponding numerators must be of the form $Ax+B$; for example,

$$\frac{3x-1}{(2x^2-1)^2(x+1)}=\frac{Ax+B}{(2x^2-1)^2}+\frac{Cx+D}{2x^2-1}+\frac{E}{x+1},$$

which gives

$$3x-1=(Ax+B)(x+1)$$
$$+(Cx+D)(x+1)(2x^2-1)+E(2x^2-1)^2.$$

For $x=-1$, this gives $E=-4$. Replacing, transposing, collecting like terms, and dividing by $x+1$, we get

$$16x^3-16x^2+3=2Cx^3+2Dx^2+x(A-C)+(B-D).$$

Hence $2C=16$ and $C=8$; $2D=-16$ and $D=-8$; $A-C=0$ or $A-8=0$ and $A=8$; and finally, $B-D=3$ or $B=-5$. So that we obtain as the partial fractions:

$$\frac{8x-5}{(2x^2-1)^2}+\frac{8(x-1)}{2x^2-1}-\frac{4}{x+1}.$$

It is useful to check the results obtained. The simplest way is to replace x by a single value, say $+1$, both in the given expression and in the partial fractions obtained.

Whenever the denominator contains but a power of a single factor, a very quick method is as follows:

Taking, for example, $\frac{4x+1}{(x+1)^3}$, let $x+1=z$; then $x=z-1$.

Replacing, we get

$$\frac{4(z-1)+1}{z^3}=\frac{4z-3}{z^3}=\frac{4}{z^2}-\frac{3}{z^3}.$$

The partial fractions are, therefore,

$$\frac{4}{(x+1)^2}-\frac{3}{(x+1)^3}.$$

Applying this to differentiation, let it be required to differentiate $y=\frac{5-4x}{6x^2+7x-3}$; we have

$$\frac{dy}{dx}=-\frac{(6x^2+7x-3)\times 4+(5-4x)(12x+7)}{(6x^2+7x-3)^2}$$

$$=\frac{24x^2-60x-23}{(6x^2+7x-3)^2}.$$

If we split the given expression into

$$\frac{1}{3x-1}-\frac{2}{2x+3},$$

we get, however,

$$\frac{dy}{dx}=-\frac{3}{(3x-1)^2}+\frac{4}{(2x+3)^2},$$

which is really the same result as above split into partial fractions. But the splitting, if done after differentiating, is more complicated, as will easily be seen. When we shall deal with the *integration* of such expressions, we shall find the splitting into partial fractions a precious auxiliary (see p. 230).

Exercises XI. (See page 293 for Answers.)

Split into fractions:

(1) $\frac{3x+5}{(x-3)(x+4)}$. (2) $\frac{3x-4}{(x-1)(x-2)}$.

(3) $\frac{3x+5}{x^2+x-12}$. (4) $\frac{x+1}{x^2-7x+12}$.

(5) $\frac{x-8}{(2x+3)(3x-2)}$. (6) $\frac{x^2-13x+26}{(x-2)(x-3)(x-4)}$.

(7) $\frac{x^2-3x+1}{(x-1)(x+2)(x-3)}$.

(8) $\frac{5x^2+7x+1}{(2x+1)(3x-2)(3x+1)}$.

(9) $\frac{x^2}{x^3-1}$.

(10) $\frac{x^4+1}{x^3+1}$.

(11) $\frac{5x^2+6x+4}{(x+1)(x^2+x+1)}$.

(12) $\frac{x}{(x-1)(x-2)^2}$.

(13) $\frac{x}{(x^2-1)(x+1)}$.

(14) $\frac{x+3}{(x+2)^2(x-1)}$.

(15) $\frac{3x^2+2x+1}{(x+2)(x^2+x+1)^2}$.

(16) $\frac{5x^2+8x-12}{(x+4)^3}$.

(17) $\frac{7x^2+9x-1}{(3x-2)^4}$.

(18) $\frac{x^2}{(x^3-8)(x-2)}$.

Differential of an Inverse Function.

Consider the function (see p. 14) $y=3x$; it can be expressed in the form $x=\frac{y}{3}$; this latter form is called the *inverse function* to the one originally given.

If $y=3x$, $\frac{dy}{dx}=3$; if $x=\frac{y}{3}$, $\frac{dx}{dy}=\frac{1}{3}$, and we see that

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}} \quad \text{or} \quad \frac{dy}{dx}\times\frac{dx}{dy}=1.$$

Consider $y=4x^2$, $\frac{dy}{dx}=8x$; the inverse function is

$$x=\frac{y^{\frac{1}{2}}}{2}, \quad \text{and} \quad \frac{dx}{dy}=\frac{1}{4\sqrt{y}}=\frac{1}{4\times 2x}=\frac{1}{8x}.$$

Here again $\frac{dy}{dx} \times \frac{dx}{dy} = 1.$

It can be shown that for all functions which can be put into the inverse form, one can always write

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

It follows that, being given a function, if it be easier to differentiate the inverse function, this may be done, and the reciprocal of the differential coefficient of the inverse function gives the differential coefficient of the given function itself.

As an example, suppose that we wish to differentiate $y = \sqrt[2]{\frac{3}{x} - 1}$. We have seen one way of doing this, by writing $u = \frac{3}{x} - 1$, and finding $\frac{dy}{du}$ and $\frac{du}{dx}$. This gives

$$\frac{dy}{dx} = -\frac{3}{2x^2\sqrt{\frac{3}{x} - 1}}.$$

If we had forgotten how to proceed by this method, or wished to check our result by some other way of obtaining the differential coefficient, or for any other reason we could not use the ordinary method, we could proceed as follows: The inverse function is $x = \frac{3}{1 + y^2}$.

$$\frac{dx}{dy} = -\frac{3 \times 2y}{(1 + y^2)^2} = -\frac{6y}{(1 + y^2)^2};$$